Name: Marco A. Montes de Oca | MATH 243 - Quiz 3
Section: 51 | March 19, 2013

Please, SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Find the tangent plane to $f(x,y) = \frac{2x+3}{4y+1}$ at $(0,0)$.

   Solution: $f_x(x,y) = \frac{2}{4y+1}$, $f_y(x,y) = (2x+3)\frac{(4y+1)(0)-(4)(1)}{(4y+1)^2} = \frac{-4(2x+3)}{(4y+1)^2}$. The equation of the tangent plane to $f(x,y)$ at $(0,0)$ is thus:

   $z - 3 = 2(x-0) - 12(y-0) = 2x - 12y$, or $2x - 12y - z + 3 = 0$.

2. (25 pts) The humidity at position (x,y,z) is modeled as $H(x,y,z) = -x^2 + y^2 - z^2$. An airplane's position after t seconds is given by $x = t^2$, $y = 3 - 4t$, and $z = \sqrt{t}$. How fast is the humidity changing on the airplane's trajectory after 9 seconds?

   Solution: $\frac{dH}{dt} = H_x(x,y,z)\frac{dx}{dt} + H_y(x,y,z)\frac{dy}{dt} + H_z(x,y,z)\frac{dz}{dt}$. So, $H_x(x,y,z) = -2x$, $H_y(x,y,z) = 2y$, and $H_z(x,y,z) = -2z$. Also, $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = -4$, and $\frac{dz}{dt} = \frac{1}{2\sqrt{t}}$.

   Evaluating everything at $t = 9$, we have: $x = 81$, $y = -33$, $z = 3$, $\frac{dx}{dt} = 18$, $\frac{dy}{dt} = -4$, $\frac{dz}{dt} = \frac{1}{6}$, $H_x(81,-33,3) = -162$, $H_y(81,-33,3) = -66$, and $H_z(81,-33,3) = -6$.

   Putting everything together, we obtain:

   $\frac{dH}{dt} = -162(18) - 66(-4) - 6(1/6) = -2653.$

3. (25 pts) Let f be a function of two variables that has continuous partial derivatives and consider the points $A(1,3)$, $B(3,3)$, $C(1,7)$, and $D(6,15)$. The directional derivative of f at A in the direction of $\vec{AB}$ is 3 and the directional derivative at A in the direction of $\vec{AC}$ is 26. Find the directional derivative of f at A in the direction of the vector $\vec{AD}$.

   Solution: $\vec{AB} = \langle 2, 0\rangle$, so $\hat{AB} = \langle 1, 0\rangle$. $\vec{AC} = \langle 0, 4\rangle$, so $\hat{AC} = \langle 0, 1\rangle$.

We know $D_{\hat{AB}}f(1,3) = \langle f_x(1,3), f_y(1,3)\rangle \cdot \langle 1,0\rangle = f_x(1,3) = 3$. Also. $D_{\hat{AC}}f(1,3) = \langle f_x(1,3), f_y(1,3)\rangle \cdot \langle 0,1\rangle = f_y(1,3) = 26$.

Now, $\vec{AD} = \langle 5,12\rangle$, so $\hat{AD} = \langle 5/13, 12/13\rangle$. So $D_{\hat{AD}}f(1,3) = \langle 3,26\rangle \cdot \langle 5/13, 12/13\rangle = 15/13 + 24 = 327/13$.

4. (25 pts) Find the local maximum and minimum values and saddle point(s) of the function $f(x,y) = x^2 + xy + y^2 + y$.

   Solution: If $\nabla f(x,y) = \langle 2x+y, 2y+x+1\rangle = 0$, then $x = 1/3$ and $y = -2/3$. The determinant of the Hessian of $f(x,y)$ is $4-1 = 3 > 0$, so the point $(1/3, -2/3)$ is the only critical point and is a local minimum.

Bonus (10 pts): Sketch the region in the xy-plane where the determinant of the Hessian of $f(x,y) = x^2 + y^3 + xy$ is negative.

Solution: The determinant of the Hessian of $f(x,y)$ is equal to $12y - 1$. So if $12y - 1 < 0$, then $y < 1/12$. The region in the xy-plane where the Hessian of $f(x,y)$ is negative is all the region $y < 1/12$.