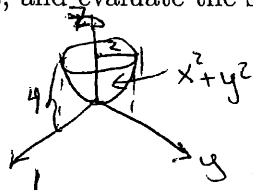
Name: MATH 243 - Quiz 5

Section: 51 April 24, 2013

Please, SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Convert the integral $\int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} x\,dz\,dy\,dx$ from rectangular coordinates to both cylindrical and spherical coordinates, and evaluate the simplest iterated integral.

The region of integration is:



Cylindrical coordinates:

$$\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r\cos\theta\,(r)\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^{2}\left(4r^2\cos\theta - r^4\cos\theta\right)dr\,d\theta =$$

$$\int_0^{2\pi}\left(\frac{4}{3}r^3\cos\theta - \frac{r^5}{5}\cos\theta\right)\Big|_0^2 d\theta =$$

$$\int_0^{2\pi}\frac{64}{15}\cos\theta\,d\theta =$$

$$\frac{64}{15}\sin\theta\Big|_0^{2\pi} = 0$$

Spherical coordinates:

z lower limit: $z = x^2+y^2 \Rightarrow \rho\cos\phi = \rho^2\sin^2\phi$
$\Rightarrow \rho = \cot\phi\csc\phi$

upper limit: $z = 4 \Rightarrow \rho\cos\phi = 4 \Rightarrow \rho = 4\sec\phi$

θ: $0 \le \theta \le 2\pi$

ϕ: lower limit $= 0$
upper limit occurs when $\rho\cos\phi = 4$ and $\rho\sin\phi = 2 \Rightarrow \tan\phi = \frac{1}{2}$
or $\phi = \arctan\frac{1}{2}$.

$$\therefore \int_0^{2\pi}\int_0^{\arctan\frac{1}{2}}\int_{\cot\phi\csc\phi}^{4\sec\phi} \rho\sin\phi\cos\theta\,(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

2. (25 pts) Find a function $f(x,y)$ such that $\nabla f(x,y) = \langle 2xy, x^2-y\rangle$.

$$\frac{\partial f_c}{\partial x} = 2xy \Rightarrow f_c = \int 2xy\,dx = x^2y + C(y)$$

$$\frac{\partial f_c}{\partial y} = x^2 + C'(y) = x^2 - y \Rightarrow C'(y) = -y \Rightarrow C(y) = -\frac{y^2}{2} + K$$

$$\therefore f(x,y) = x^2y - \frac{y^2}{2} + K$$

3. (25 pts) Use the FTC4LIs to find the work done by the force field $\vec{F}(x,y) = \langle 2xy, x^2 - y \rangle$ on a particle as it moves from the point $(1,2)$ to $(4,0)$ along a straight line. (Note that the field in this problem is equal to the gradient of the previous exercise.)

By the FTC4LIs:

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(t_f)) - f(\vec{r}(t_i)) =$$

$$f(\langle 4,0 \rangle) - f(\langle 1,2 \rangle) =$$

$$(4)^2(0) - \frac{0^2}{2} + K - \left((1)^2(2) - \frac{(2)^2}{2} + K\right) =$$

$$K - (2 - 2 + K) = \underline{0}$$

4. (25 pts) Evaluate $\int_C y\,dx + x^2\,dy$ where C is the parabolic arc given by $y = 4x - x^2$ from $(4,0)$ to $(1,3)$.

$$dy = (4-2x)dx \quad \int_C y\,dx + x^2\,dy = \int_4^1 (4x - x^2)dx + x^2(4-2x)dx =$$

$$\int_4^1 (4x - x^2 + 4x^2 - 2x^3)\,dx = \int_4^1 (-2x^3 + 3x^2 + 4x)\,dx =$$

$$-\frac{2}{4}x^4 + x^3 + 2x^2\Big]_4^1 = \left(-\frac{1}{2} + 1 + 2\right) - (-128 + 64 + 32) =$$

$$\frac{5}{2} - (-32) = \boxed{\frac{69}{2}}$$

Bonus (10 pts): If $\vec{F}(x,y,z) = \langle a_1, a_2, a_3 \rangle$ is a constant force vector field, show that the work done in moving a particle along any path from P to Q is $W = \vec{F} \cdot \vec{PQ}$.

Since $\nabla \times \vec{F} = \vec{0}$, there is an f such that $\nabla f = \vec{F}$.
One such f is $f(x,y,z) = a_1 x + a_2 y + a_3 z$.

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(Q) - f(P) = a_1 x_q + a_2 y_q + a_3 z_q -$$

$$a_1 x_p + a_2 y_p + a_3 z_p =$$

$$a_1(x_q - x_p) + a_2(y_q - y_p) + a_3(z_q - z_p)$$

$$= \vec{F} \cdot \vec{PQ}$$