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Section: 51

MATH 243 - Quiz 6

May 13, 2013

Please, SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Find the surface area of the part of the paraboloid $z = x^2 + y^2$ over the unit circle.
2. (25 pts) Evaluate the surface integral $\iint_S (y^2 + 2yz)\, dS$, where S is the portion of the plane $2x + y + 2z = 6$ in the first octant.
3. (25 pts) Find the flux of $\boldsymbol{F}(x, y, z) = \langle x, y, z\rangle$ through S, where S is the portion of the paraboloid $z = 4 - x^2 - y^2$ lying above the xy-plane with upward orientation.
4. (25 pts) Use the Divergence Theorem to find the flux of $\boldsymbol{F}(x, y, z) = \langle 2z, y, y^2\rangle$ through a sphere centered at the origin with radius 2 and outward orientation.

Bonus (10 pts): Describe (with words and/or with a sketch) the surface whose parametric representation is $\boldsymbol{r}(u, v) = \langle 5\cos(u), 5\cos(u), v\rangle$, where $0 \le u \le 2\pi$, and $0 \le v \le 2$.

1. Parametric representation of paraboloid:

$\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$

$\vec{r}_x(x,y) = \langle 1, 0, 2x \rangle$; $\vec{r}_y(x,y) = \langle 0, 1, 2y \rangle$

$$\vec{n} = \vec{r}_x(x,y) \times \vec{r}_y(x,y) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

$\|\vec{n}\| = \sqrt{4x^2 + 4y^2 + 1}$

$$SA = \iint_D \sqrt{4x^2+4y^2+1}\, dA \overset{\text{polar}}{\Rightarrow} SA = \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\, r\,dr\,d\theta$$

$u = 4r^2+1$; $du = 8r\,dr$

$$SA = \frac{1}{8}\int_0^{2\pi}\int_1^5 \sqrt{u}\, du\, d\theta = \frac{1}{8}\int_0^{2\pi} \frac{2}{3}u^{3/2}\Big|_1^5 \, d\theta = \underline{\frac{\pi}{6}(5^{3/2}-1)}$$

2. $S: 2x+y+2z=6 \Rightarrow$

$$\vec{r}(x,y) = \left\langle x, y, \frac{6-2x-y}{2}\right\rangle$$

$$\vec{r}_x(x,y) = \langle 1, 0, -1\rangle \ ; \ \vec{r}_y(x,y) = \langle 0, 1, -\tfrac{1}{2}\rangle$$

$$\vec{n} = \vec{r}_x(x,y) \times \vec{r}_y(x,y) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} = \langle 1, \tfrac{1}{2}, 1\rangle$$

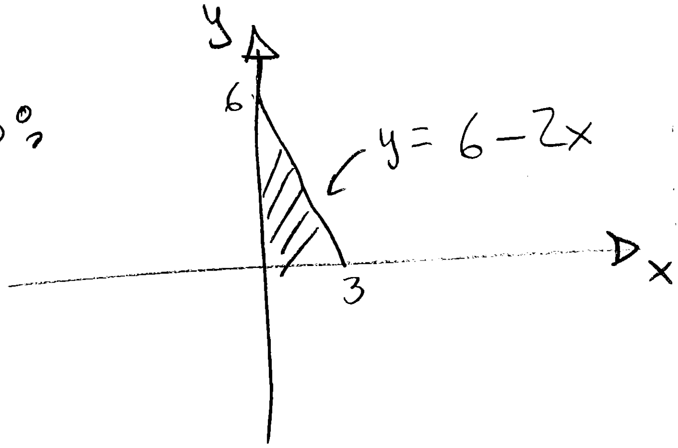
$$\|\vec{n}\| = \sqrt{1+\frac{1}{4}+1} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

$\therefore$

$$\iint_S (y^2+2yz)\,dS = \iint_D \left(y^2 + 2y\left(\frac{6-2x-y}{2}\right)\right)\cdot\left(\frac{3}{2}\right) dA$$

$$= \frac{3}{2}\iint_D (y^2 + 6y - 2xy - y^2)\, dA = \frac{3}{2}\iint_D (6y - 2xy)\, dA$$

D:



$$\frac{3}{2}\iint_D (6y - 2xy)\, dA = \frac{3}{2}\int_0^3\int_0^{6-2x} (6y - 2xy)\, dy\, dx =$$

$$\frac{3}{2}\int_0^3 3y^2 - xy^2 \Big|_0^{6-2x} dx = \frac{3}{2}\int_0^3 \left[3(6-2x)^2 - x(6-2x)^2\right] dx =$$

multiplying and dividing by 2

$$\frac{3}{2}\int_0^3 (6-2x)^2(3-x)\, dx = \frac{3}{2}\,\frac{1}{2}\int_0^3 (6-2x)^2(6-2x)\, dx =$$

$$\frac{3}{4}\int_0^3 (6-2x)^3\, dx = \begin{Bmatrix} u = 6-2x \\ du = -2dx \end{Bmatrix} = -\frac{3}{8}\int_6^0 u^3\, du = \frac{3}{8}\,\frac{u^4}{4}\Big|_0^6 =$$

$$\boxed{\frac{3}{32}6^4 = \frac{243}{2}}$$

3. $\vec{r}(x,y) = \langle x, y, 4-x^2-y^2 \rangle$

$\vec{r}_x(x,y) = \langle 1, 0, -2x \rangle$

$\vec{r}_y(x,y) = \langle 0, 1, -2y \rangle$

$$\vec{n} = \vec{r}_x(x,y) \times \vec{r}_y(x,y) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = 2x\,\hat{i} + 2y\,\hat{j} + 1\,\hat{k}$$

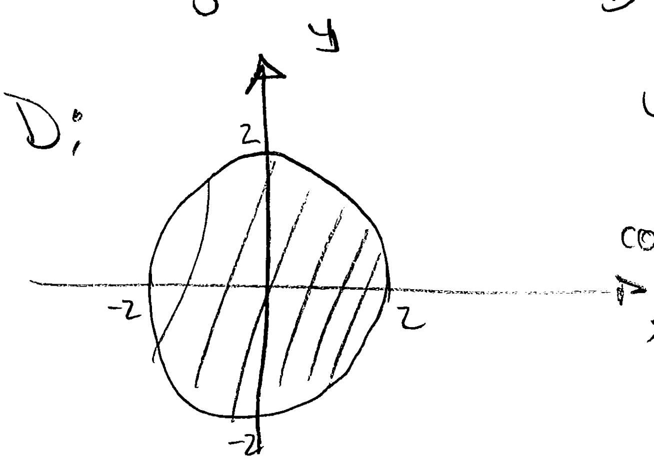
$\vec{F}(\vec{r}(x,y)) = \langle x, y, 4-x^2-y^2 \rangle$

$\therefore\ \vec{F}(\vec{r}(x,y)) \cdot \vec{n} = \langle x, y, 4-x^2-y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle =$

$2x^2 + 2y^2 + 4 - x^2 - y^2 = x^2 + y^2 + 4$

$\therefore\ \text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D (x^2+y^2+4)\, dA$, where

D:



using polar coordinates: $\text{Flux} = \int_0^{2\pi}\int_0^2 (r^2+4)\, r\, dr\, d\theta$

$$\text{Flux} = \frac{1}{2}\int_0^{2\pi}\int_4^8 u\,du\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{u^2}{2}\Big|_4^8 d\theta = \frac{1}{4}\int_0^{2\pi}(64-16)\,d\theta =$$

$u = r^2+4$
$du = 2r\,dr$

$$\text{Flux} = 12\int_0^{2\pi} d\theta = \boxed{24\pi}$$

4. $$\text{Flux} = \iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV,$$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle 2z, y, y^2\rangle$$

$$= 0+1+0 = 1$$

$$\therefore \text{Flux} = \iiint_E dV = \underbrace{\frac{4}{3}\pi(2)^3}_{\text{Volume of sphere of radius 2}} = \frac{32}{3}\pi$$

Bonus: It is a cylinder centered at the origin with radius 5 and height 2.