Exam # 1

Math 529 - Section 10

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1. $f(x_1, x_2) = x_1^2 + x_2^3$, $g(x_1, x_2) = x_1^2 + x_2^4$

$$\nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 \\ 3x_2^2 \end{pmatrix} = \vec{0} \Rightarrow x_1 = 0 \ \& \ x_2 = 0$$

$$\nabla g(x_1, x_2) = \begin{pmatrix} 2x_1 \\ 4x_2^3 \end{pmatrix} = \vec{0} \Rightarrow x_1 = 0 \ \& \ x_2 = 0$$

$$Hf(x_1, x_2) = \begin{pmatrix} 2 & 0 \\ 0 & 6x_2 \end{pmatrix}, \quad Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$Hg(x_1, x_2) = \begin{pmatrix} 2 & 0 \\ 0 & 12x_2^2 \end{pmatrix}, \quad Hg(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

We have shown that $(0,0)$ is a critical point for f and g. Also, the Hessians of f and g at $(0,0)$ are both positive semidefinite because their eigenvalues are $\{2,0\}$.

However, note that by simply letting $x_2 \to -\infty$ we can make $f \to -\infty$ too, thus $(0,0)$ is not a local minimizer for f. In contrast, no matter what step we take from $(0,0)$, we have that $g(0,0) < g(x_1, x_2)$ $x_1, x_2 \neq 0$ $\forall x_1, x_2 \in \mathbb{R}$, so $(0,0)$ is not only a local minimizer for g, it is a strict global minimizer for g.

2. $f(x,y) = x^4 + y^4 - 32y^2$

a) $$\nabla f(x,y) = \begin{pmatrix} 4x^3 \\ 4y^3 - 64y \end{pmatrix}$$

$$Hf(x,y) = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 - 64 \end{pmatrix}$$

Since the eigenvalues of $Hf(x,y) = \{12x^2, 12y^2-64\}$, it is enough to make $12x^2 > 0$ and $12y^2-64 < 0$ to have a point (x,y) at which $Hf(x,y)$ is indefinite.

For $12x^2 > 0$, we just need to pick $x > 0$, say $x = 1$.

For $12y^2 - 64 < 0$, we have $12y^2 < 64 \Rightarrow$

$$y^2 < \frac{16}{3} \Rightarrow$$

$$|y| < \sqrt{\frac{16}{3}} = \frac{4}{\sqrt{3}}$$

So, if $y = 1$, $12y^2 - 64 < 0$.

Thus, at $(1,1)$ $Hf(\cancel{x},1)$ is indefinite.

b) To minimize $f(x,y)$, let $\nabla f(x,y) = \vec{0}$, so we have

(1) $4x^3 = 0 \Rightarrow x = 0$

(2) $4y^3 - 64y = y(4y^2-64) = 0 \Rightarrow y = 0$, or $4y^2 - 64 = 0 \Rightarrow$

$$y^2 = 16 \Rightarrow$$

$$y = \pm 4$$

So, the critical points are $(0,-4), (0,0), (0,4)$.

@ $(0,-4)$

$$Hf(0,-4) = \begin{pmatrix} 0 & 0 \\ 0 & 368 \end{pmatrix}, \quad \text{eigenvalues} = \{0, 368\}$$

so $(0,-4)$ is a local minimizer

@ $(0,0)$ we fall into the range for Hf to be indefinite. Therefore, $(0,0)$ is a saddle point.

@ $(0,4)$

$$Hf(0,4) = \begin{pmatrix} 0 & 0 \\ 0 & 368 \end{pmatrix}, \quad \text{eigenvalues} = \{0, 368\}$$

so $(0,4)$ is a local minimizer.

3. If $f(x)$ is strictly convex, then for two distinct points in $\mathbb{R}^n$, we have

$$f(\alpha u + (1-\alpha)v) < \alpha f(u) + (1-\alpha) f(v)$$

with $0 < \alpha < 1$.

Therefore, since $f(u) = f(v) = 0$, then

$$f(\alpha u + (1-\alpha)v) < 0$$

It suffices then to pick $z \in [u,v]$ (that is in the line joining u with v) to have

$$f(z) < 0.$$

4. Let us analyze the Hessian of f to determine what kind of function f is.

$$\nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 + x_2 - 1 \\ x_1 + 4x_2 - 4 \end{pmatrix}$$

$$Hf(x_1, x_2) = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

Since all the leading principal minors are positive, we can conclude that $Hf(x_1, x_2)$ is positive definite. Therefore, $f(x_1, x_2)$ is strictly convex.

5. Newton's method iteration is:

$$x_{k+1} = x_k - Hf_k^{-1} \nabla f_k$$

So, we have:

$$\nabla f(x_1, x_2) = \begin{pmatrix} 8x_1^3 - 4x_2 \\ 2x_2 - 4x_1 + 5 \end{pmatrix}$$

$$Hf(x_1, x_2) = \begin{pmatrix} 24x_1^2 & -4 \\ -4 & 2 \end{pmatrix}$$

Since $x_0^T = (0,0)$, $\nabla f_0 = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$, $Hf_0 = \begin{pmatrix} 0 & -4 \\ -4 & 2 \end{pmatrix}$

and $Hf_0^{-1} = \frac{1}{-16} \begin{pmatrix} 2 & 4 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{8} & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{pmatrix}$

So,

$$x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -\frac{1}{8} & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \end{pmatrix} = -\begin{pmatrix} -\frac{5}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} \\ 0 \end{pmatrix}$$

Now, $\nabla f_1 = \begin{pmatrix} 8\left(\frac{5}{4}\right)^3 - 0 \\ 0 - 4\left(\frac{5}{4}\right) + 5 \end{pmatrix} = \begin{pmatrix} \frac{250}{16} \\ 0 \end{pmatrix}$

$$Hf_1 = \begin{pmatrix} 24\left(\frac{5}{4}\right)^2 & -4 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{75}{2} & -4 \\ -4 & 2 \end{pmatrix}$$

$$Hf_1^{-1} = \frac{1}{59}\begin{pmatrix} 2 & 4 \\ 4 & \frac{75}{2} \end{pmatrix} = \begin{pmatrix} \frac{2}{59} & \frac{4}{59} \\ \frac{4}{59} & \frac{75}{118} \end{pmatrix}$$

So

$$X_2 = X_1 - Hf_1^{-1}\nabla f_1 = \begin{pmatrix} \frac{5}{4} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{59} & \frac{4}{59} \\ \frac{4}{59} & \frac{75}{118} \end{pmatrix}\begin{pmatrix} \frac{250}{16} \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{5}{4} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{59} \cdot \frac{250}{16} \\ \frac{4}{59} \cdot \frac{250}{16} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{5}{4} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{250}{472} \\ \frac{250}{236} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} - \frac{250}{472} \\ -\frac{250}{236} \end{pmatrix} =$$

$$\begin{pmatrix} 0.7203 \\ -1.0593 \end{pmatrix}$$

Bonus: $f(x_1, x_2) = (x_1 + x_2^2)^2$

$$\nabla f(x_1, x_2) = \begin{pmatrix} 2(x_1 + x_2^2)(1) \\ 2(x_1 + x_2^2)(2x_2) \end{pmatrix} = $$

$$= \begin{pmatrix} 2(x_1 + x_2^2) \\ 4x_2(x_1 + x_2^2) \end{pmatrix}$$

$$\nabla f(1,0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$\nabla f(1,0) \cdot p < 0$ for p to be a descent direction, so

$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -2 < 0$, so p is indeed a descent direction.