Exam # 2

Math 529 - Section 10

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1. $\{x_k\} = \{e^{-k}\}$.

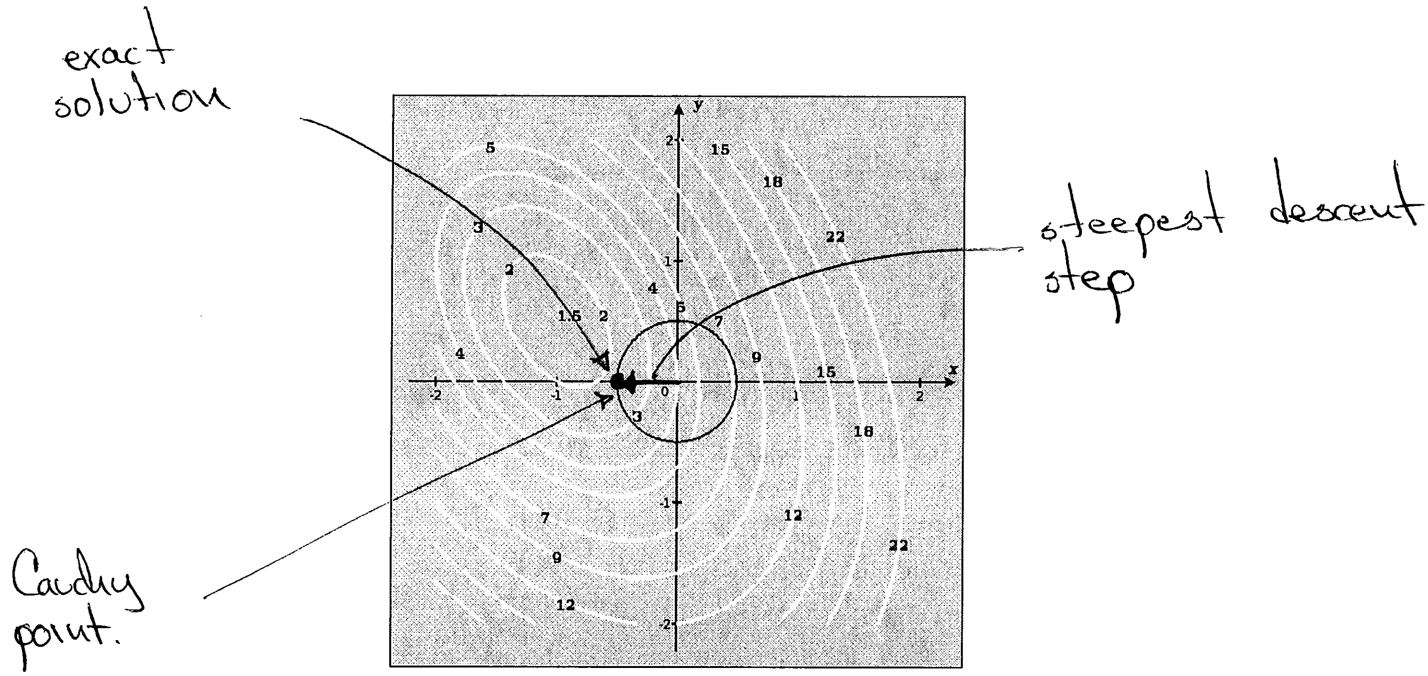
This sequence converges to zero because

$$\lim_{k \to \infty} e^{-k} = 0$$

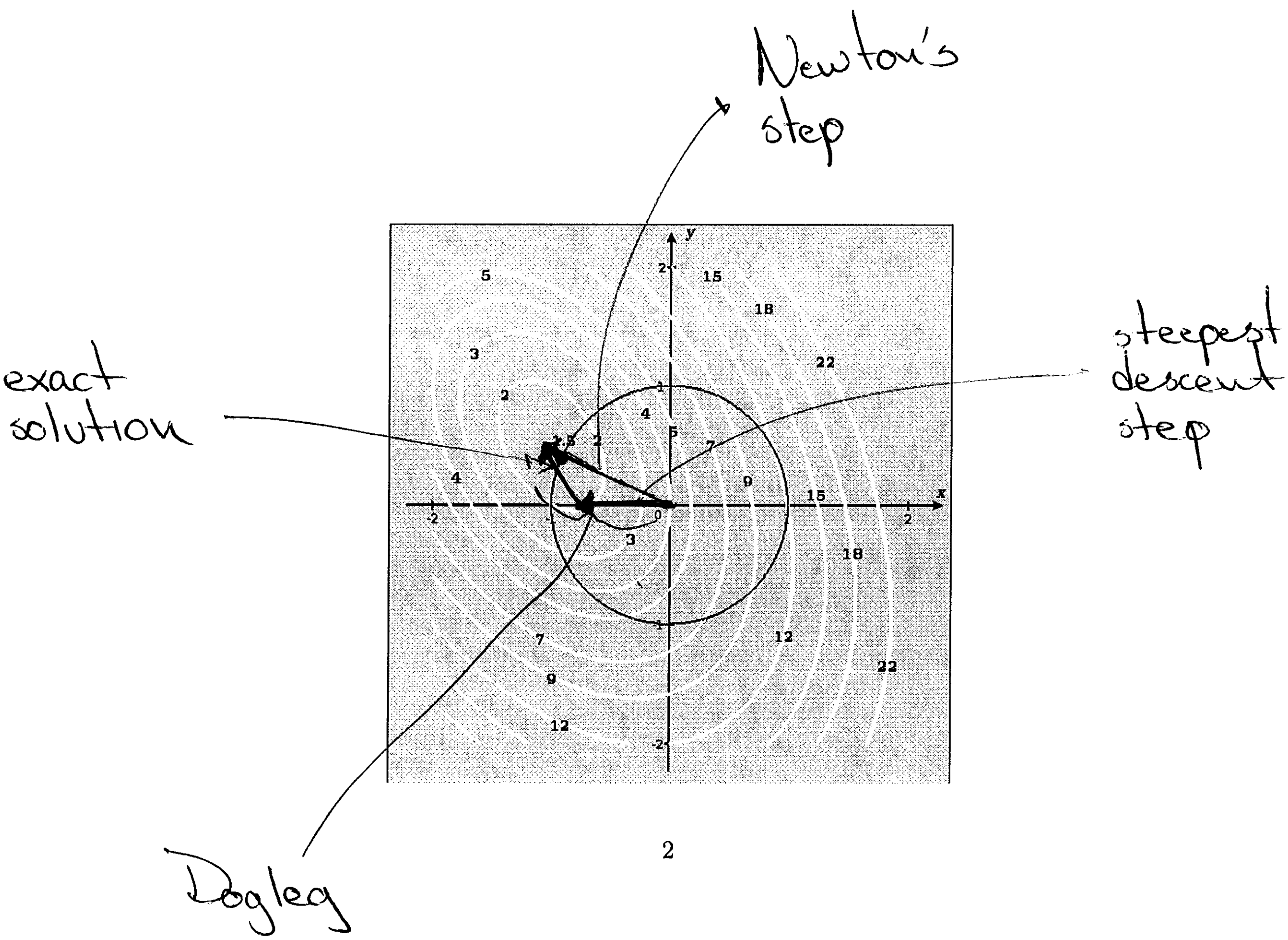
The rate of convergence is:

$$\frac{|x_{k+1} - 0|}{|x_k - 0|} = \frac{e^{-(k+1)}}{e^{-k}} = e^{-1} = c \text{ (a constant)}$$

Therefore, the rate of convergence of $\{x_k\}$ is q-linear. It is not q-superlinear because $\lim_{k \to \infty} e^{-1} = e^{-1} \neq 0$ and it is not q-quadratic because $\frac{e^{-(k+1)}}{(e^{-k})^2} = e^{k-1} \neq c$ (a constant)


exact solution
steepest descent step
Cauchy point.
y
x
5
15
18
3
22
2
4
1.5
2
7
9
15
3
18
7
12
22
9
12

a) $\Delta_k = 0.5$


Newton's step
exact solution
steepest descent step
Dogleg
y
x
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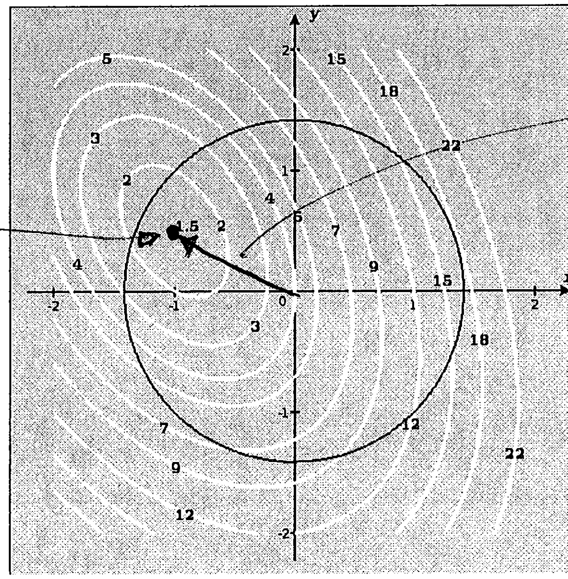
b) $\Delta_k = 1$

exact solution



Newton's step

c) $\Delta_k = 1.5$

3. $z = x^2 + y^2$, subject to $x + 4y = 2$

$$L(x,y,\lambda) = x^2 + y^2 + \lambda(2 - x - 4y)$$

$$\nabla L(x,y,\lambda) = \begin{pmatrix} 2x - \lambda \\ 2y - 4\lambda \\ 2 - x - 4y \end{pmatrix} = \vec{0}$$

(1) $2x - \lambda = 0 \Rightarrow x = \frac{\lambda}{2}$

(2) $2y - 4\lambda = 0 \Rightarrow y = \frac{4\lambda}{2} = 2\lambda$

(3) $2 - x - 4y = 0$

$$2 - \left(\frac{\lambda}{2}\right) - 4(2\lambda) = 0$$

$$2 - \frac{\lambda}{2} - 8\lambda = 0$$

$$2 - \frac{17\lambda}{2} = 0 \Rightarrow 4 = 17\lambda \Rightarrow \lambda = \frac{4}{17}$$

$$\therefore\ x = \frac{\frac{4}{17}}{2} = \frac{2}{17}$$

$$y = 2\left(\frac{4}{17}\right) = \frac{8}{17}$$

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & 4 \\ 1 & 2 & 0 \\ 4 & 0 & 2 \end{vmatrix} = -1(2-0) + 4(0-8) = -2-32 = -34$$

Since this is a 2 variable, 1 constraint problem, the fact that $|\bar{H}| < 0$ means $\left(\frac{2}{17}, \frac{8}{17}\right)$ is a minimizer.

4. Let a point on the circle be denoted by (x,y) and let a point on the line be denoted by (u,v). The distance between these points is $d(x,y,u,v) = \sqrt{(x-u)^2 + (y-v)^2}$. The minimum of $d(x,y,u,v)$ is the same as $d^2(x,y,u,v) = D$, but D is easier to work with, so we are going to work with D instead of d.

The optimization problem is therefore:

minimize $D(x,y,u,v) = (x-u)^2 + (y-v)^2$

subject to $x^2+y^2=1$

$u+v=4$

$$L(x,y,u,v,\lambda,\mu) = (x-u)^2 + (y-v)^2 + \lambda(1-x^2-y^2) + \mu(4-u-v)$$

$$L(x,y,u,v,\lambda,\mu) = \begin{pmatrix} 2(x-u)-2\lambda x \\ 2(y-v)-2\lambda y \\ -2(x-u)-\mu \\ -2(y-v)-\mu \\ 1-x^2-y^2 \\ 4-u-v \end{pmatrix} = \vec{0}$$

From (1), (2), (3) & (4)

$$\left.\begin{aligned} -\mu &= 2(x-u) = 2\lambda x \\ -\mu &= 2(y-v) = 2\lambda y \end{aligned}\right\} \therefore \; 2\lambda x = 2\lambda y \Rightarrow x = y$$

because $\lambda \neq 0$ as there is no intersection between the line and the circle.

Substituting y in (5):

$$2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

so $\quad y = \pm\frac{1}{\sqrt{2}}$

From (1) & (2):

$$u = x(1-\lambda)$$
$$v = y(1-\lambda)$$

Substituting the previous result and u, v in 6:

$$x(1-\lambda) + y(1-\lambda) = 4 \quad \text{but since } x = y:$$

$$2x(1-\lambda) = 4 \Rightarrow$$

$$1-\lambda = \frac{4}{2x} = \frac{2}{x} = \frac{2}{\pm\frac{1}{\sqrt{2}}} = \pm 2\sqrt{2}$$

$$\Rightarrow \lambda = 1 \mp 2\sqrt{2}$$

Thus

$$u = \pm\frac{1}{\sqrt{2}}\left(1 - (1 \mp 2\sqrt{2})\right)$$
$$= \pm\frac{1}{\sqrt{2}}(\pm 2\sqrt{2}) = \pm 2$$

$v = \pm 2$, but $(u=-2, v=-2)$ violates (6), so the only solution is $(u=2, v=2)$.

Finally, $\mu = -2\lambda x = -2(1 \mp 2\sqrt{2})\left(\pm\frac{1}{\sqrt{2}}\right)$
$$= \mp\frac{2}{\sqrt{2}}(1 \mp 2\sqrt{2})$$

Therefore, the critical points are

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2, 2\right) \quad \text{and}$$

$$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 2, 2\right)$$

of these, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2, 2\right)$ is clearly the minimizer and the minimum distance is

$$d = \sqrt{\left(\frac{1}{\sqrt{2}} - 2\right)^2 + \left(\frac{1}{\sqrt{2}} - 2\right)^2} = \sqrt{2\left(\frac{1}{\sqrt{2}} - 2\right)^2}$$

$$= \sqrt{2\left(\frac{1-2\sqrt{2}}{\sqrt{2}}\right)^2} = \sqrt{\cancel{2}\,\frac{(1-2\sqrt{2})^2}{\cancel{2}}} = |1 - 2\sqrt{2}|$$

The bordered Hessian for this problem is

$$|\bar{H}| = \begin{vmatrix} 0 & 0 & 2x & 2y & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 2x & 0 & 2-2\lambda & 0 & -2 & 0 \\ 2y & 0 & 0 & 2-2\lambda & 0 & -2 \\ 0 & 1 & -2 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{vmatrix}$$

5. maximize $100x + 40y$

subject to $30x + 20y = 1000$

$80x + 20y = 2000$

$$L(x,y,\lambda,\mu) = 100x + 40y + \lambda(1000 - 30x - 20y) + \mu(2000 - 80x - 20y)$$

$$\nabla L(x,y,\lambda,\mu) = \begin{pmatrix} 100 - 30\lambda - 80\mu \\ 40 - 20\lambda - 20\mu \\ 1000 - 30x - 20y \\ 2000 - 80x - 20y \end{pmatrix} = \vec{0}$$

From (1) and (2)

$$\lambda = \frac{6}{5} \quad \text{and} \quad \mu = \frac{4}{5}$$

$\therefore$ Since $\lambda > \mu$, it is expected that increasing the budget would give more profit than increasing the amount of thread.

Bonus:

$$g = \nabla f(x_0) = \begin{pmatrix} 16x_{01} - 4x_{01}^3 \\ 16x_{02} - 4x_{02}^3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 16 - 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

$$B = Hf(x_0) = \begin{pmatrix} 16 - 12x_{01}^2 & 0 \\ 0 & 16 - 12x_{02}^2 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 0 \\ 0 & 4 \end{pmatrix}$$

$$M(p) = 6 + g^T p + \frac{1}{2} p^T B p$$

$$= 6 + 12p_2 + 8p_1^2 + 2p_2^2$$

$$\nabla m(p) = \begin{pmatrix} 16 p_1 \\ 12 + 4 p_2 \end{pmatrix} = \vec{0}$$

$\therefore \quad p_1 = 0 \quad p_2 = -3$

This point is a minimizer because B is positive definite.