Homework #1

Math 529 - Section 10

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1. Not sure you solve optimization problems in your everyday life? Keep thinking.

2. Report.

3. Report.

4. a) $f(x) = x^2 + 2x$ on $\mathbb{R}$.

$f'(x) = 2x + 2$, so if x^* is a critical point, then

$f'(x^*) = 2x^* + 2 = 0 \Rightarrow x^* = -1$

Since $f''(x) = 2 > 0$ for all $x \in \mathbb{R}$

$x^* = -1$ is a strict global minimizer for $f(x)$.

x^* is also a local minimizer by definition.

$f(x)$ has no local or global maximizers.

b) $f(x) = x^4 + 4x^3 + 6x^2 + 4x$ on $\mathbb{R}$.

$f'(x) = 4x^3 + 12x^2 + 12x + 4$.

If x^* is a critical point, then $f'(x^*) = 0$

Or (after dividing by 4):

$$x^{*3} + 3x^{*2} + 3x^* + 1 = 0$$

$$(x^*+1)^3 = 0 \Rightarrow x^* = -1$$

$f''(x) = 3(x+1)^2 \geq 0$ for all $x \in \mathbb{R}$

Therefore, since $f''(x) = 0$ only when $x = x^*$, we can conclude that $x^* = -1$ is a strict global minimizer for $f(x)$.

No local or global maximizers exist for $f(x)$.

c) $f(x) = x + \sin x$ on $\mathbb{R}$

$f'(x) = 1 + \cos x \Rightarrow$ if x^* is a critical point then

$f'(x^*) = 0 \Rightarrow x^* = n\pi$, where $n = \ldots, -5, -3, -1, 1, 3, 5, \ldots$

Now, $f''(x) = -\sin x$ which is positive or negative for different values of x; therefore $f(x)$ has no global minimizers or maximizers.

Also, since $f''(x^*) = 0$, $f(x)$ has no local minimizers or maximizers. In fact x^* are inflection points only.

5. a) positive definite
   b) negative definite
   c) indefinite
   d) positive definite
   e) negative definite
   f) indefinite

6.

a) $-x_1^2 + 3x_2^2 + 4x_1x_2$

b) $2x_1^2 - 6x_1x_2$

c) $x_1^2 - 2x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_2x_3$

d) $-3x_1^2 + 2x_2^2 + 4x_3^2 + 2x_1x_2 + 4x_1x_3 - 2x_2x_3$

7. a)

$$\vec{x} \cdot \begin{pmatrix} 9 & 1 \\ 1 & -3 \end{pmatrix} \vec{x}$$

b)

$$\vec{x} \cdot \begin{pmatrix} 1 & -2 & 3 \\ -2 & -3 & 1 \\ 3 & 1 & 4 \end{pmatrix} \vec{x}$$

c)

$$\vec{x} \cdot \begin{pmatrix} 3 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \vec{x}$$

8.

a) $f(x_1,x_2) = x_1^2 - 4x_1 + 2x_2^2 + 7$

$$\nabla f(x_1,x_2) = \begin{pmatrix} 2x_1 - 4 \\ 4x_2 \end{pmatrix}$$

If $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ is a critical point, then

$$\nabla f(x_1^*, x_2^*) = \vec{0} \Rightarrow \vec{x}^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$Hf(x_1,x_2) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

Since $Hf(x_1,x_2)$ is positive definite for all $\vec{x} \in \mathbb{R}^2$, $\vec{x}^*$ is a strict global minimizer of f. f does not have a global maximizer in $\mathbb{R}^2$.

b) $f(x_1,x_2) = e^{-(x_1^2+x_2^2)}$

$$\nabla f(\vec{x}) = \begin{pmatrix} -2x_1 e^{-(x_1^2+x_2^2)} \\ -2x_2 e^{-(x_1^2+x_2^2)} \end{pmatrix}$$

If $\vec{x}^*$ is a critical point, then

$$\nabla f(\vec{x}^*) = \vec{0} \Rightarrow \vec{x}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ is the only critical point.

$$Hf(\vec{x}) = \begin{pmatrix} e^{-(x_1^2+x_2^2)}(4x_1^2-2) & e^{-(x_1^2+x_2^2)}(4x_1x_2) \\ e^{-(x_1^2+x_2^2)}(4x_1x_2) & e^{-x_1^2+x_2^2}(4x_2^2-2) \end{pmatrix}$$

The eigenvalues of $Hf(\vec{x})$ are:

$$\lambda_1 = -2e^{-(x_1^2+\cancel{x}_2^2)}$$

$$\lambda_2 = (4x_1^2+4x_2^2-2)e^{-(x_1^2+x_2^2)}$$

$\lambda_1 < 0$ for all $\vec{x} \in \mathbb{R}^2$; however,

$\lambda_2 > 0$ if $x_1^2 + x_2^2 > \frac{1}{2}$ and

$\lambda_2 < 0$ if $x_1^2 + x_2^2 < \frac{1}{2}$

This means that $Hf(\vec{x})$ is an indefinite matrix in $\mathbb{R}^2$ and therefore, by the theorems discussed in class, we cannot automatically conclude anything about $\vec{x}^*$.

However note that

$$f(\vec{x}) = e^{-\|\vec{x}\|^2},$$ where $\|\vec{x}\|$ is the Euclidean norm of $\vec{x}$.

So, $$\lim_{\|\vec{x}\| \to \infty} f(\vec{x}) = 0$$

and since $\|\vec{x}^*\| = \|\vec{0}\| = 0$ and therefore, we may conclude that $\vec{x}^*$ is a strict global maximizer of f, even though we could not detect it with the 2nd order conditions.

c) $f(\vec{x}) = x_1^2 - 2x_1x_2 + \frac{1}{3}x_2^3 - 4x_2$

$$\nabla f(\vec{x}) = \begin{pmatrix} 2x_1 - 2x_2 \\ -2x_1 + x_2^2 - 4 \end{pmatrix} \Rightarrow \text{if } \nabla f(\vec{x}^*) = \vec{0}$$

$$\Rightarrow x_1^* = x_2^* \Rightarrow x_2^{*2} - 2x_2^* - 4 = 0 \Rightarrow x_2^* = x_1^* = \begin{cases} 1+\sqrt{5} \\ 1-\sqrt{5} \end{cases}$$

Now, $$Hf(\vec{x}) = \begin{pmatrix} 2 & -2 \\ -2 & 2x_2 \end{pmatrix},$$ which means that $Hf(\vec{x})$ is positive definite if $x_2 > 1$

So, in general, $Hf(\vec{x})$ is indefinite on $\mathbb{R}^2$.

Since $f(\vec{x})$ is actually unbounded, we may conclude that $f(\vec{x})$ does not hava any global minimizers or maximizers.

d) $f(\vec{x}) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$

$$\nabla f(\vec{x}) = \begin{pmatrix} 8x_1 - 4x_2 \\ -4x_1 + 4x_2 - 2x_3 \\ -2x_2 + 4x_3 - 2 \end{pmatrix}$$

Thus, $\vec{x}^* = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$ is the only critical point.

$$Hf(\vec{x}) = \begin{pmatrix} 8 & -4 & 0 \\ -4 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}, \quad \begin{aligned} \Delta_1 &= 8 > 0 \\ \Delta_2 &= 32 - 16 > 0 \\ \Delta_3 &= 32 > 0 \end{aligned}$$

$\therefore$ $Hf(\vec{x})$ is positive definite everywhere in $\mathbb{R}^3$ and $\vec{x}^*$ is a strict global minimizer of $f(\vec{x})$.

e) $f(\vec{x}) = x_1^4 + 16x_1x_2 + x_2^8$

$$\nabla f(\vec{x}) = \begin{pmatrix} 4x_1^3 + 16x_2 \\ 16x_1 + 8x_2^7 \end{pmatrix}$$

If $\nabla f(\vec{x}^*) = \vec{0}$, then

$$4x_1^3 = -16x_2 \Rightarrow x_1^3 = -4x_2 \quad (1)$$

$$16x_1 = -8x_2^7 \Rightarrow x_1 = -\frac{1}{2}x_2^7 \quad (2)$$

(2) in (1):

$$\left(-\frac{1}{2}x_2^7\right)^3 = -4x_2$$

$$-\frac{1}{8}x_2^{21} = -4x_2 \Rightarrow -\frac{1}{8}x_2^{21} + 4x_2 = 0$$

$$x_2\left(-\frac{1}{8}x_2^{20} + 4\right) = 0$$

$\Rightarrow x_2 = 0$ or $\frac{1}{8}x_2^{20} = 4 \Rightarrow x_2^{20} = 32$

$$|x_2| = (32)^{1/20} = 2^{5/20} = \underline{2^{1/4}}$$

$$\downarrow$$

$$\underline{x_2 = \pm 2^{1/4}}$$

and $x_1 = 0$ or $x_1 = -2^{-1}(\pm 2^{1/4})^7$

$$= \mp 2^{-1}2^{7/4} = \underline{\mp 2^{3/4}}$$

$$Hf(\vec{x}) = \begin{pmatrix} 12x_1^2 & 16 \\ 16 & 56x_2^6 \end{pmatrix}$$

Since $\Delta_2 = \det(Hf\,\vec{x}) = 672x_1^2x_2^6 - 256$ can be positive or negative depending on the values of x_1 or x_2, $Hf(\vec{x})$ is indefinite in $\mathbb{R}^2$ and the 2nd order condition cannot be used to classify the critical points in any global sense.

However, locally we can see that $\vec{x}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a saddle point because $\det(Hf(\vec{0})) < 0$.

Similarly, $\vec{x}^* = \begin{pmatrix} \pm 2^{1/4} \\ \mp 2^{3/4} \end{pmatrix}$ are local minima because at those points $Hf(\vec{x}^*)$ is positive definite.

9. a)

$$f_1(\vec{x}) = \vec{a} \cdot \vec{x} = \sum_{i=1}^{n} a_i x_i$$

$$\nabla f_1(\vec{x}) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \vec{a}$$

$$Hf_1(\vec{x}) = 0$$

b)

$$f_2(\vec{x}) = \vec{x} \cdot A\vec{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

$$\nabla f_2(\vec{x}) = \begin{pmatrix} \sum_{j=1}^{n} A_{1j} x_j + \sum_{i=1}^{n} A_{i1} x_i \\ \sum_{j=1}^{n} A_{2j} x_j + \sum_{i=1}^{n} A_{i2} x_i \\ \vdots \\ \sum_{j=1}^{n} A_{nj} x_j + \sum_{i=1}^{n} A_{in} x_i \end{pmatrix}$$

$$\nabla f_2(\vec{x}) = \begin{pmatrix} 2\sum_{j=1}^{n} A_{1j}\, x_j \\ 2\sum_{j=1}^{n} A_{2j}\, x_j \\ \vdots \\ 2\sum_{j=1}^{n} A_{nj}\, x_j \end{pmatrix}$$

because A is symmetric

$$= 2A\vec{x}$$

$$Hf(\vec{x}) = 2A$$

10. If $A = B^T B$, then $\vec{x} \cdot A\vec{x}$

$= \vec{x}^T A \vec{x} = \vec{x}^T B^T B \vec{x} = (B\vec{x})^T B\vec{x}$.

If $B\vec{x} = \vec{y}$, then

$(B\vec{x})^T B\vec{x} = \vec{y}^T \vec{y} = \vec{y} \cdot \vec{y} = \|\vec{y}\| \geq 0$

$\therefore$ A is positive semidefinite.