Homework # 4
Math 529 - 10
Marco A. Montes de Oca
Spring 2013

1. $\min_{x} \; f(x) = x_1 + x_2 \quad (1)$

s.t. $x_1^2 + x_2^2 = 1 \quad (2)$

From (2) $X_2 = \pm\sqrt{1 - x_1^2}$

Substituting $x_2 = \sqrt{1-x_1^2}$ in f:

$$f(x_1) = x_1 + \sqrt{1 - x_1^2}$$

$f'(x_1) = 1 - \dfrac{x_1}{\sqrt{1-x_1^2}}$; if $f'(x_1) = 0$, then

$$\frac{\sqrt{1-x_1^2} - x_1}{\sqrt{1-x_1^2}} = 0 \Rightarrow \sqrt{1-x_1^2} = x_1$$

$$1 - x_1^2 = x_1^2$$

$$1 = 2x_1^2$$

$$x_1 = \pm \frac{1}{\sqrt{2}}$$

Therefore

$$x_2 = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Solution 1: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Now, if $x_2 = -\sqrt{1 - x_1^2}$, $x_1 = \pm \frac{1}{\sqrt{2}}$, but now

$$x_2 = -\frac{1}{\sqrt{2}}$$

Solution 2: $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

The correct answer is $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$; however, if we use $x_2 = \sqrt{1 - x_1^2}$, we miss the solution.

2. $L(x, \lambda) = x_1 + x_2 + \lambda(1 - x_1^2 - x_2^2)$

$$\nabla L(x, \lambda) = \begin{pmatrix} 1 - 2\lambda x_1 \\ 1 - 2\lambda x_2 \\ 1 - x_1^2 - x_2^2 \end{pmatrix} = \vec{0}$$

So,

$$x_1 = \frac{1}{2\lambda}$$

$$x_2 = \frac{1}{2\lambda}$$

and

$$1 - \left(\frac{1}{2\lambda}\right)^2 - \left(\frac{1}{2\lambda}\right)^2 = 0$$

$$1 = 2\left(\frac{1}{2\lambda}\right)^2 = 2\left(\frac{1}{4\lambda^2}\right) = \frac{1}{2\lambda^2} \Rightarrow \lambda^2 = \frac{1}{2}$$

$$\lambda = \pm\frac{1}{\sqrt{2}} \therefore x_1 = \pm\frac{\sqrt{2}}{2} \;\&\; x_2 = \pm\frac{\sqrt{2}}{2}$$

Solutions: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$$|\bar{H}| = \begin{vmatrix} 0 & -2x_1 & -2x_2 \\ -2x_1 & -2\lambda & 0 \\ -2x_2 & 0 & -2\lambda \end{vmatrix}$$

@ $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$|\bar{H}| = \begin{vmatrix} 0 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & \frac{-2}{\sqrt{2}} & 0 \\ -\sqrt{2} & 0 & -\frac{2}{\sqrt{2}} \end{vmatrix} = \sqrt{2}(2) - \sqrt{2}(-2) = 4\sqrt{2} > 0$$

$\therefore \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is a local maximum.

@ $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$$|\bar{H}| = \begin{vmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \frac{2}{\sqrt{2}} & 0 \\ \sqrt{2} & 0 & \frac{2}{\sqrt{2}} \end{vmatrix} = -\sqrt{2}(2) + \sqrt{2}(-2) = -4\sqrt{2} < 0$$

$\therefore \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is a local minimum.

3. $f(x,y) = (x-1)^2 + (y-2)^2$

s.t. $(x-1)^2 = 5y$

$$L(x,y,\lambda) = (x-1)^2 + (y-2)^2 + \lambda\left(5y - (x-1)^2\right)$$

$$\nabla L(x,y,\lambda) = \begin{pmatrix} 2(x-1) - 2\lambda(x-1) \\ 2(y-2) + 5\lambda \\ 5y - (x-1)^2 \end{pmatrix} = \vec{0}$$

$$x - 1 - \lambda x + \lambda = 0 \qquad (1)$$

$$2y + 5\lambda - 4 = 0 \qquad (2)$$

$$5y - x^2 + 2x - 1 = 0 \qquad (3)$$

From (1):

$$x(1-\lambda) - (1-\lambda) = 0 \Rightarrow x = 1 \quad (\text{if } \lambda \neq 1)$$

In (3):

$$5y - 1 + 2 - 1 = 0 \Rightarrow y = 0$$

In (2):

$$5\lambda - 4 = 0 \Rightarrow \lambda = \frac{4}{5}$$

Solution $(1,0)$ $\lambda = \frac{4}{5}$

If $\lambda = 1$:

From (2)

$$2y + 5 - 4 = 0 \Rightarrow y = -\frac{1}{2}$$

In (3)

$$-\frac{5}{2} - x^2 + 2x - 1 = 0$$

$$x^2 - 2x = -\frac{7}{2}$$

$$x^2 - 2x + \frac{7}{2} = 0$$

, which does not have real solutions.

$\therefore$ The only solution is $(1, 0)$

Part c) By substituting $y = \frac{(x-1)^2}{5}$ into $f(x,y)$:

$$f(x) = (x-1)^2 + \left[\frac{(x-1)^2}{5} - 2\right]^2$$

$$= (x-1)^2 + \frac{1}{25}(x-1)^4 - \frac{4}{5}(x-1)^2 + 4$$

$$= \frac{1}{5}(x-1)^2 + \frac{1}{25}(x-1)^4 + 4$$

$$f'(x) = \frac{2}{5}(x-1) + \frac{4}{25}(x-1)^3 = 0$$

$$= \frac{2}{5}(x-1)\left(1 + \frac{2}{5}(x-1)^2\right) = 0$$

So,

$$\frac{2}{5}(x-1) = 0 \quad \text{or}$$

$$\frac{2}{5}(x-1)^2 + 1 = 0$$

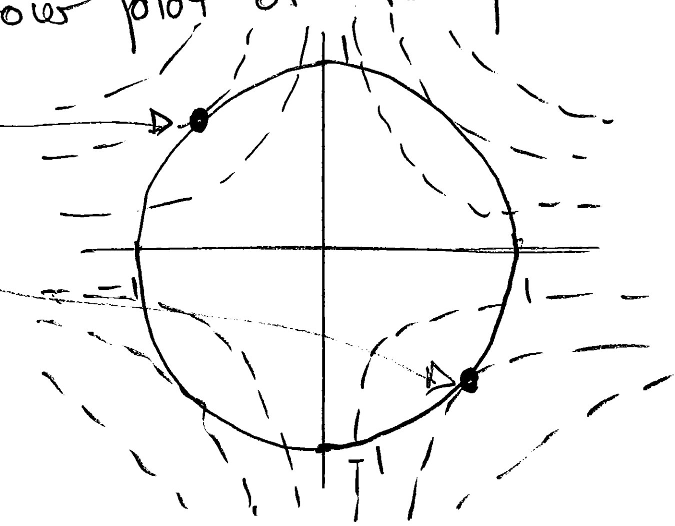
If $\frac{2}{5}(x-1) = 0 \Rightarrow x = 1$, and $y = 0$

If $\frac{2}{5}(x-1)^2 + 1 = 0$, implies

$2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$, but there is no solution for x, given $y = -\frac{1}{2}$.

4. The contour plot of the problem is roughly

MINIMA



$$L(x, \lambda) = x_1 x_2 + \lambda(1 - x_1^2 - x_2^2)$$

$$\nabla L(x,\lambda) = \begin{pmatrix} x_2 - 2\lambda x_1 \\ x_1 - 2\lambda x_2 \\ 1 - x_1^2 - x_2^2 \end{pmatrix} = \vec{0}$$

$x_2 - 2\lambda x_1 = 0$ (1) $\Rightarrow x_2 = 2\lambda x_1$

$x_1 - 2\lambda x_2 = 0$ (2) $\Rightarrow x_1 = 2\lambda(2\lambda x_1) = 4\lambda^2 x_1 \Rightarrow$

$1 - x_1^2 - x_2^2 = 0$ (3) $\quad \lambda^2 = \frac{1}{4} \Rightarrow \lambda = \pm\frac{1}{2}$

if $x_1 \neq 0$

Then, $x_2 = \pm 2\left(\frac{1}{2}\right) x_1 = \pm x_1$

In (3):

$$1 - x_1^2 - (\pm x_1)^2 = 0$$

$$1 - x_1^2 - x_1^2 = 0$$

$$1 - 2x_1^2 = 0 \Rightarrow x_1^2 = \frac{1}{2} \Rightarrow x_1 = \pm\frac{1}{\sqrt{2}}$$

so $x_2 = \mp \frac{1}{\sqrt{2}}$

If $x_1 = 0$, the system is inconsistent, so x_1 cannot be equal to zero.

$$|\bar{H}| = \begin{vmatrix} 0 & 2x_1 & 2x_2 \\ 2x_1 & -2\lambda & 1 \\ 2x_2 & 1 & -2\lambda \end{vmatrix}$$

@ $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ $\lambda = -\frac{1}{2}$

$$|\bar{H}| = \begin{vmatrix} 0 & \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & 1 & 1 \\ -\frac{2}{\sqrt{2}} & 1 & 1 \end{vmatrix} = -\frac{2}{\sqrt{2}}\left(\frac{2}{\sqrt{2}} - \left(-\frac{2}{\sqrt{2}}\right)\right) - \frac{2}{\sqrt{2}}\left(\frac{2}{\sqrt{2}} - \left(-\frac{2}{\sqrt{2}}\right)\right) =$$

$$-\frac{8}{\sqrt{2}} - \frac{8}{\sqrt{2}} = -\frac{16}{\sqrt{2}} < 0$$

So, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is a local minimizer.

@ $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\lambda = \frac{1}{2}$

$$|\bar{H}| = \begin{vmatrix} 0 & -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ -\frac{2}{\sqrt{2}} & -1 & 1 \\ \frac{2}{\sqrt{2}} & 1 & -1 \end{vmatrix} = \frac{2}{\sqrt{2}}\left(\frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}}\right) + \frac{2}{\sqrt{2}}\left(-\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right)$$

$= 0$ So, the determinantal test does not provide information

Yet $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a minimizer.

So $\max f(x,y,z) = xyz$

s.t $x+y+z=S$

$$L(x,y,z,\lambda) = xyz + \lambda(S-x-y-z)$$

$$\nabla L(x,y,z,\lambda) = \begin{pmatrix} yz - \lambda \\ xz - \lambda \\ xy - \lambda \\ S-x-y-z \end{pmatrix} = \vec{0}$$

So, $yz = \lambda = xz$

$yz = \lambda = xy$

Since $x,y,z > 0$, we can divide by any of them, therefore

$$\begin{matrix} y = x \\ z = x \end{matrix} \Rightarrow x = y = z = \frac{S}{3}$$

Thus, any product $xyz \leq \left(\frac{S}{3}\right)^3 \Rightarrow$

$$\sqrt[3]{xyz} \leq \frac{S}{3} = \frac{x+y+z}{3}$$

$$\min f(x,y,z) = xy + 3xz + 7yz$$
$$\text{s.t.} \quad 3y = x$$
$$xyz = 12$$

$$L(x,y,z,\lambda,\mu) = xy + 3xz + 7yz + \lambda(x-3y) + \mu(12-xyz)$$

$$\nabla L(x,y,z,\lambda,\mu) = \begin{pmatrix} y + 3z + \lambda - \mu yz \\ x + 7z - 3\lambda - \mu xz \\ 3x + 7y - \mu xy \\ x - 3y \\ 12 - xyz \end{pmatrix} \begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \\ (5) \end{array} = \vec{0}$$

From (4) $x = 3y$, so

(1) $y + 3z + \lambda - \mu yz = 0$

(2) $3y + 7z - 3\lambda - 3\mu yz = 0$

(3) $9y + 7y - 3\mu y^2 = 0$

(5) $12 - 3y^2 z = 0$

From (5) $z = \dfrac{12}{3y^2} = \dfrac{4}{y^2}$ $(y \neq 0)$, so

(1) $y + \frac{12}{y^2} + \lambda - \mu y\left(\frac{4}{y^2}\right) = 0$

$y + \frac{12}{y^2} + \lambda - \frac{4\mu}{y} = 0$

(2) $3y + 7\left(\frac{4}{y^2}\right) - 3\lambda - 3\mu y\left(\frac{4}{y^2}\right) = 0$

$3y + \frac{28}{y^2} - 3\lambda - \frac{12\mu}{y} = 0$

(4) $16y - 3\mu y^2 = 0$

(1) & (2) times y^2:

(1) $y^3 + 12 + \lambda y^2 - 4\mu y = 0$

(2) $3y^3 + 28 - 3\lambda y^2 - 12\mu y = 0$

(4) $16y - 3\mu y^2 = 0$

From (4) $y(16 - 3\mu y) = 0$ (since we assumed $y \neq 0$, we have to check $16 - 3\mu y = 0$, only)

$16 - 3\mu y = 0 \Rightarrow \mu = \frac{16}{3y}$, thus

(1)

$$y^3 + 12 + \lambda y^2 - 4\left(\frac{16}{3y}\right) y = 0$$

$$y^3 + 12 + \lambda y^2 - \frac{64}{3} = 0$$

$$y^3 + \lambda y^2 - \frac{28}{3} = 0 \quad \Rightarrow \quad \lambda = \frac{\frac{28}{3} - y^3}{y^2}$$

(2)

$$3y^3 + 28 - 3\lambda y^2 - 64 = 0$$

$$3y^3 - 3\lambda y^2 - 36 = 0 \quad (\div 3)$$

$$y^3 - \lambda y^2 - 12 = 0$$

$$y^3 - \left(\frac{\frac{28}{3} - y^3}{y^2}\right) y^2 - 12 = 0$$

$$y^3 - \frac{28}{3} + y^3 - 12 = 0$$

$$2y^3 = \frac{28}{3} + 12 = \frac{64}{3}$$

$$y^3 = \frac{32}{3} \Rightarrow y \approx 2.201$$

Therefore, $x \approx 3(2.201) = 6.603$ $\quad \mu \approx 2.423$

$\lambda \approx -0.274$

$$z \approx \frac{12}{(2.201)(6.603)} = 0.8254$$

The bordered Hessian is

$$|\bar{H}| = \begin{vmatrix} 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & yz & xz & xy \\ -1 & yz & 0 & 1-\mu z & 3-\mu y \\ 3 & xz & 1-\mu z & 0 & 7-\mu x \\ 0 & xy & 3-\mu y & 7-\mu x & 0 \end{vmatrix}$$

at $(6.603, 2.201, 0.8254)$ with $\lambda = -0.274$, $\mu = 2.423$

$$|\bar{H}| = \begin{vmatrix} 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1.815 & 5.447 & 14.533 \\ -1 & 1.815 & 0 & -1 & -2.333 \\ 3 & 5.447 & -1 & 0 & -9 \\ 0 & 14.533 & -2.333 & -9 & 0 \end{vmatrix}$$

Here $n=3$ (# of variables), $K=2$ (# of constraints), so we have to check $n-K=3-2=1$ leading principal minors, or $|\bar{H}|$ itself.

$|\bar{H}|=3797.8>0$, which has the sign $(-1)^K=(-1)^2=+1$. Therefore $(6.603, 2.201, 0.8254)$ is a minimizer.