Homework # 5

Math 529 - Section 10

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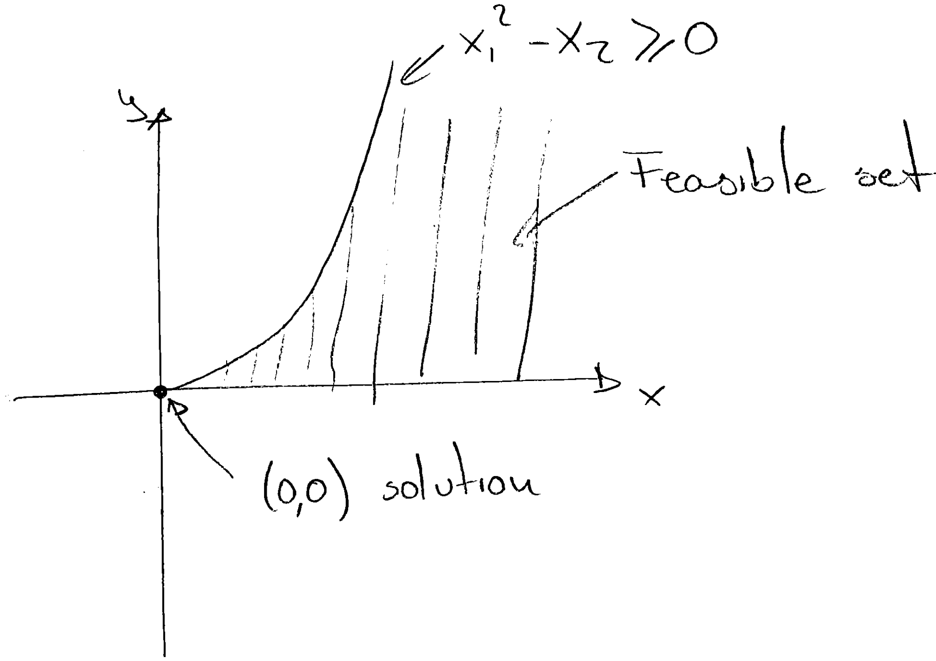
1.



LICQ:

$$\nabla c_1(x_1, x_2) = \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} \quad @ (0,0) \quad \nabla c_1(0,0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\nabla c_2(x_1, x_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \nabla c_2(0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\nabla c_3(x_1, x_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \nabla c_3(0,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Clearly, the vectors $\nabla c_1(0,0)$, $\nabla c_2(0,0)$, and $\nabla c_3(0,0)$ are not linearly independent. Therefore LICQ is not satisfied at $(0,0)$.

MFCQ:

If MFCQ is satisfied at $(0,0)$, then there should be a vector $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ such that

$$a^T \nabla c_1(0,0) > 0,$$
$$a^T \nabla c_2(0,0) > 0, \text{ and}$$
$$a^T \nabla c_3(0,0) > 0$$

$$a^T \nabla c_1(0,0) = -a_2 \qquad (1)$$
$$a^T \nabla c_2(0,0) = a_1 \qquad (2)$$
$$a^T \nabla c_3(0,0) = a_2 \qquad (3)$$

From (1) and (3) it should be clear that there is no value for a_2 that could satisfy $-a_2 > 0$ & $a_2 > 0$ simultaneously. Therefore, MFCQ is not satisfied at $(0,0)$.

KKT:

$$L(x,\lambda) = x_1 + \lambda_1(0 - x_1^2 + x_2) + \lambda_2(0 - x_1) + \lambda_3(0 - x_2)$$

$$= x_1 - \lambda_1(x_1^2 - x_2) - \lambda_2 x_1 - \lambda_3 x_2$$

$$\nabla_x L(x,\lambda) = \begin{pmatrix} 1 - 2\lambda_1 x_1 - \lambda_2 \\ \lambda_1 - \lambda_3 \end{pmatrix} = 0$$

Therefore, the KKT conditions for this problem are:

$1 - 2\lambda_1 x_1 - \lambda_2 = 0$ (1)

$\lambda_1 - \lambda_3 = 0$ (2)

$x_1^2 - x_2 \geq 0$ (3)

$x_1 \geq 0$ (4)

$x_2 \geq 0$ (5)

$\lambda_1(x_1^2 - x_2) = 0$ (6)

$\lambda_2 x_1 = 0$ (7)

$\lambda_3 x_2 = 0$ (8)

$\lambda_1, \lambda_2, \lambda_3 \geq 0$ (9)

At $(0,0)$, (3), (4), (5), (6), (7), (8) are satisfied, but we have to determine whether $\lambda_1, \lambda_2, \lambda_3 \geq 0$.

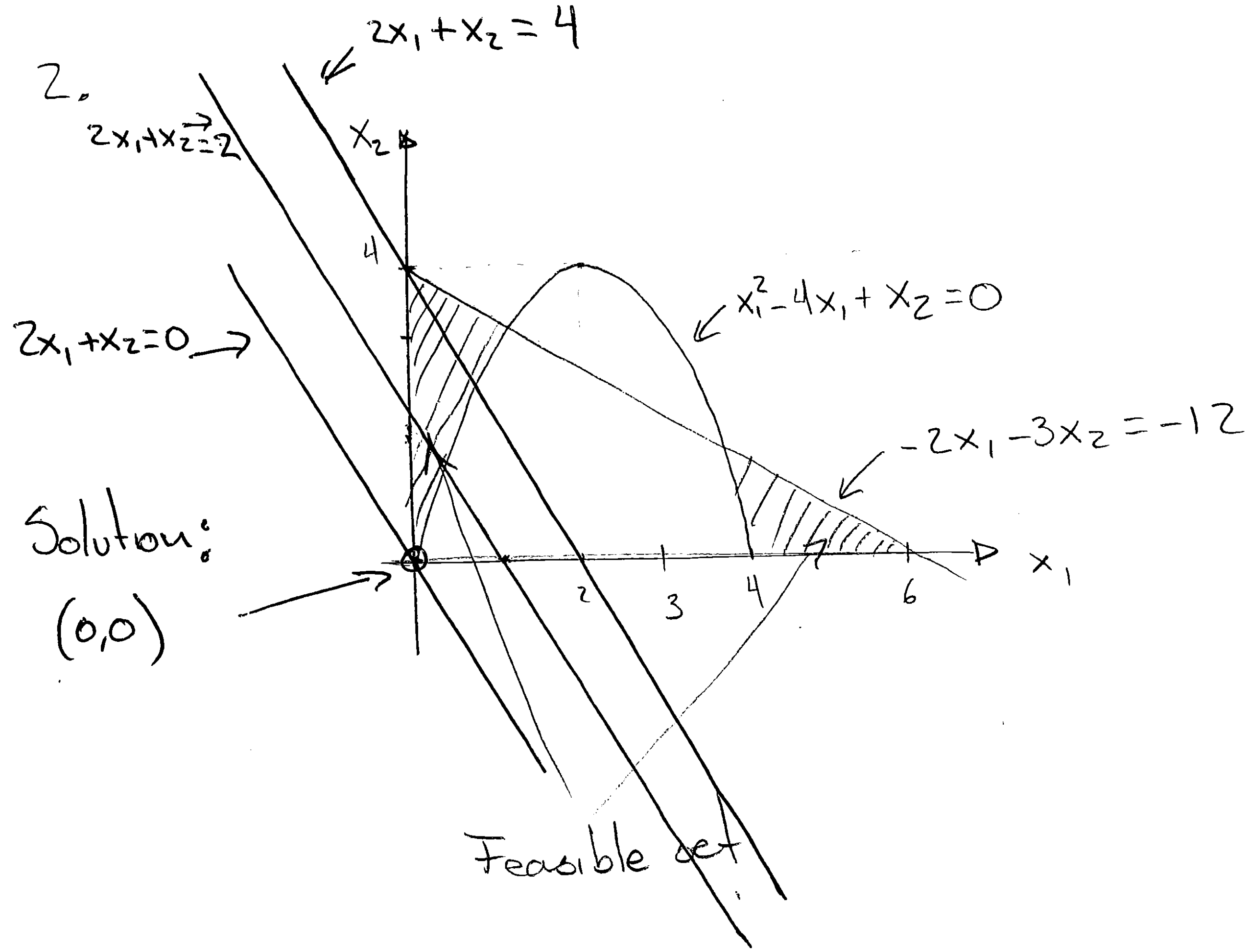
From ① and assuming $x_1 = 0$, we have that

$1 - \lambda_2 = 0 \Rightarrow \lambda_2 = 1 \geq 0$.

From ②, we have that $\lambda_1 - \lambda_3 = 0 \Rightarrow \lambda_1 = \lambda_3$.

So, any values $\lambda_1 \geq 0$, $\lambda_3 \geq 0$ would satisfy the KKT conditions as long as $\lambda_1 = \lambda_3$.

2.



LICQ: At the solution, the active constraints are c_1, c_3 and c_4. So,

$$\nabla c_1(x) = \begin{pmatrix} 2x_1 - 4 \\ 1 \end{pmatrix} \quad @ (0,0) \quad \nabla c_1(0,0) = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\nabla c_3(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \nabla c_4(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

As in the previous case, $\nabla c_1(0,0)$, $\nabla c_3(0,0)$ & $\nabla c_4(0,0)$ are not linearly independent, so LICQ is not satisfied at $(0,0)$.

MFCQ: For MFCQ to be satisfied, we need to find $\vec{a}$ such that

$$a^T \nabla c_1(0,0) > 0$$
$$a^T \nabla c_3(0,0) > 0 \; \&$$
$$a^T \nabla c_4(0,0) > 0$$

If $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, we have

$$a^T \nabla c_1(0,0) = -4a_1 + a_2$$
$$a^T \nabla c_3(0,0) = a_1$$
$$a^T \nabla c_4(0,0) = a_2$$

Therefore we need

$-4a_1 + a_2 > 0 \Rightarrow a_2 > 4a_1$

$a_1 > 0$ $\therefore$ $\vec{a}$ can be $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ So MFCQ is satisfied.

$a_2 > 0$

KKT:

$$L(x,\lambda) = 2x_1 + x_2 + \lambda_1(0 - x_1^2 + 4x_1 - x_2) + \lambda_2(-12 + 2x_1 + 3x_2) +$$
$$\lambda_3(0 - x_1) + \lambda_4(0 - x_2)$$

$$= 2x_1 + x_2 - \lambda_1(x_1^2 - 4x_1 + x_2) - \lambda_2(12 - 2x_1 - 3x_1) -$$
$$\lambda_3(x_1) - \lambda_4(x_2)$$

$$\nabla_x L(x,\lambda) = \begin{pmatrix} 2 - 2\lambda_1 x_1 + 4\lambda_1 + 2\lambda_2 - \lambda_3 \\ 1 - \lambda_1 + 3\lambda_2 - \lambda_4 \end{pmatrix} = 0$$

So the KKT conditions for this problem are:

$2 - 2\lambda_1 x_1 + 4\lambda_1 + 2\lambda_2 - \lambda_3 = 0$ (1)

$1 - \lambda_1 + 3\lambda_2 - \lambda_4 = 0$ (2)

$x_1^2 - 4x_1 + x_2 \geq 0$ (3)

$-2x_1 - 3x_2 \geq -12$ (4)

$x_1, x_2 \geq 0$ (5)

$\lambda_1(x_1^2 - 4x_1 + x_2) = 0$ (6)

$\lambda_2(-12 + 2x_1 + 3x_2) = 0$ (7)

$\lambda_3 x_1 = 0$ (8)

$\lambda_4 x_2 = 0$ (9)

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ (10)

when $x_1 = 0$, $x_2 = 0$:

From (7): $\lambda_2 = 0$

Then

(1) becomes $2 + 4\lambda_1 - \lambda_3 = 0$ (1')

(2) becomes $1 - \lambda_1 - \lambda_4 = 0$ (2')

There are 8 possible combinations of values for λ_1, λ_3 and λ_4:

	λ_1	λ_3	λ_4	
	0	0	0	Violates (1') and (2')
	0	0	>0	Violates (1')
	0	>0	0	Violates (2')
	0	>0	>0	Does not violate any condition, so the KKT conditions are valid with $x_1 = 0$, $x_2 = 0$, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 2$, $\lambda_4 = 1$
Not needed	>0	0	0	
	>0	0	>0	
	>0	>0	0	
	>0	>0	>0	

3. $f(x) = x_1^2 x_2^2 + x_1^2 - x_1$

$$\nabla f(x) = \begin{pmatrix} 2x_1 x_2^2 + 2x_1 - 1 \\ 2x_1^2 x_2 \end{pmatrix}$$

$$Hf(x) = \begin{pmatrix} 2x_2^2 + 2 & 4x_1 x_2 \\ 4x_1 x_2 & 2x_1^2 \end{pmatrix}$$

@ $(1,0)$

$$\nabla f(1,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad Hf(1,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Using ∇f and Hf for $m_k(p)$, we have:

$$m_k(p) = f(1,0) + p^T \nabla f(1,0) + \tfrac{1}{2} p^T Hf(1,0) p$$

$$= 0 + p^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} p^T \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} p$$

$$= p_1 + p_1^2 + p_2^2$$

The exact solution of the subproblem requires us to solve:

$$\min_{p} \quad p_1^2 + p_2^2 + p_1$$

$$\text{subject to} \quad \|p\| = \sqrt{p_1^2 + p_2^2} \leq \frac{1}{4}$$

or

$$p_1^2 + p_2^2 \leq \frac{1}{16}$$

$$L(p,\lambda) = p_1^2 + p_2^2 + p_1 + \lambda_1\left(-\frac{1}{16} + p_1^2 + p_2^2\right)$$

$$\nabla_p L(p,\lambda) = \begin{pmatrix} 2p_1 + 1 + 2\lambda_1 p_1 \\ 2p_2 + 2\lambda_1 p_2 \end{pmatrix} = 0$$

So the KKT conditions are:

$$2p_1 + 1 + 2\lambda_1 p_1 = 0 \quad (1)$$

$$2p_2 + 2\lambda_1 p_2 = 0 \quad (2)$$

$$p_1^2 + p_2^2 \leq \frac{1}{16} \quad (3)$$

$$\lambda_1\left(-\frac{1}{16} + p_1^2 + p_2^2\right) = 0 \quad (4)$$

$$\lambda_1 \geq 0 \quad (5)$$

If $\lambda_1 = 0$, we have:

$$\left.\begin{aligned} 2p_1 + 1 = 0 &\Rightarrow p_1 = -\frac{1}{2} \\ 2p_2 = 0 &\Rightarrow p_2 = 0 \end{aligned}\right\} \text{ Violates } ③$$

If $\lambda_1 > 0$, we have:

ⓐ $$p_1^2 + p_2^2 = \frac{1}{16}$$

ⓑ $$2p_1 + 1 + 2\lambda_1 p_1 = 0$$

ⓒ $$2p_2 + 2\lambda_1 p_2 = 0 \Rightarrow 2p_2(1+\lambda_1) = 0 \Rightarrow$$

$$\underbrace{p_2 = 0}_{\text{only possibility}} \quad \text{or} \quad \underbrace{\lambda_1 = -1}_{\text{violates } ⑤}$$

In ⓐ

$$\therefore \; p_1^2 = \frac{1}{16} \Rightarrow p_1 = \pm\frac{1}{4}$$

If $p_1 = \frac{1}{4}$, in ⓑ

$$2\left(\frac{1}{4}\right) + 1 + \frac{2\lambda_1}{4} = 0$$

$$\frac{3}{2} + \frac{\lambda_1}{2} = 0 \Rightarrow \underbrace{\lambda_1 = -3}_{\text{violates } ⑤}$$

So $p_1 = -\frac{1}{4}$ $p_2 = 0$ and $\lambda_1 = 3$ is the only solution to the KKT conditions

Cauchy point:

$$\nabla m_k(0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \text{the steepest descent}$$

direction is $-\nabla m(0,0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. However, since $\|\nabla m_k(0,0)\| = 1 > \Delta_k = \frac{1}{4}$, then the Cauchy point is calculated as follows:

$$p_k^C = -\Delta_k \frac{\nabla m_k}{\|\nabla m_k\|} = -\frac{1}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ 0 \end{pmatrix}$$

which corresponds to the exact solution to the subproblem using the KKT conditions.

4. Evaluating ∇f and Hf at $(1,1)$:

$$\nabla f(1,1) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad ; \quad Hf(1,1) = \begin{pmatrix} 4 & 4 \\ 4 & 2 \end{pmatrix}$$

Since $\det(Hf(1,1)) < 0$ $Hf(1,1)$ is an indefinite matrix.

The eigenvalues of $Hf(x)$ are -1.1231 and 7.1231

Adjusting Hf so that we work with a positive definite matrix:

$$B = Hf + \underbrace{1.2}(I) = \begin{pmatrix} 5.2 & 4 \\ 4 & 3.2 \end{pmatrix}$$

$$\downarrow$$

$|\mu| < 1.2$, where μ is the smallest eigenvalue.

Using $\nabla f(1,1)$ and B for $m_k(p)$, we have

$$m_k(p) = f(1,1) + p^T \nabla f(1,1) + \frac{1}{2} p^T B p$$

$$= 1 + p^T \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \frac{1}{2} p^T \begin{pmatrix} 5.2 & 4 \\ 4 & 3.2 \end{pmatrix} p =$$

$$= 1 + 3p_1 + 2p_2 + 2.6 p_1^2 + 1.6 p_2^2 + 4 p_1 p_2$$

The subproblem is now:

$$\min_p \; 2.6 p_1^2 + 1.6 p_2^2 + 4 p_1 p_2 + 3p_1 + 2p_2 + 1$$

subject to

$$p_1^2 + p_2^2 \leq \frac{1}{16}$$

$$L(p,\lambda) = 2.6p_1^2 + 1.6p_2^2 + 4p_1p_2 + 3p_1 + 2p_2 + 1 + \lambda\left(-\frac{1}{16} + p_1^2 + p_2^2\right)$$

$$\nabla_p L(p,\lambda) = \begin{pmatrix} 5.2p_1 + 4p_2 + 3 + 2\lambda p_1 \\ 3.2p_2 + 4p_1 + 2 + 2\lambda p_2 \end{pmatrix} = 0$$

The KKT conditions are:

$$5.2p_1 + 4p_2 + 3 + 2\lambda p_1 = 0 \quad (1)$$

$$3.2p_2 + 4p_1 + 2 + 2\lambda p_2 = 0 \quad (2)$$

$$p_1^2 + p_2^2 \leq \frac{1}{16} \quad (3)$$

$$\lambda\left(-\frac{1}{16} + p_1^2 + p_2^2\right) = 0 \quad (4)$$

$$\lambda \geq 0 \quad (5)$$

The solution that satisfies these conditions is $p_1 = -0.2203$, $p_2 = -0.1181$ and $\lambda = 3.135$

Cauchy point:

$$\nabla M_k(p) = \begin{pmatrix} 5.2p_1 + 4p_2 + 3 \\ 3.2p_2 + 4p_1 + 2 \end{pmatrix} ; \quad \nabla M_k(0,0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Since $\|\nabla m_k(0,0)\| > \frac{1}{4}$, then

$$p_k^c = -\Delta_k \frac{\nabla m_k(0,0)}{\|\nabla m_k(0,0)\|} = -\frac{1}{4}\begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix} = \begin{pmatrix} -\frac{3}{4\sqrt{13}} \\ -\frac{1}{2\sqrt{13}} \end{pmatrix}$$

or $p_k^c = \begin{pmatrix} -0.2080 \\ -0.1386 \end{pmatrix}$

The reduction obtained with the exact solution is

from $m_k(0,0) = 1$ to $m_k(-0.2203, -0.1181) = 0.355469$

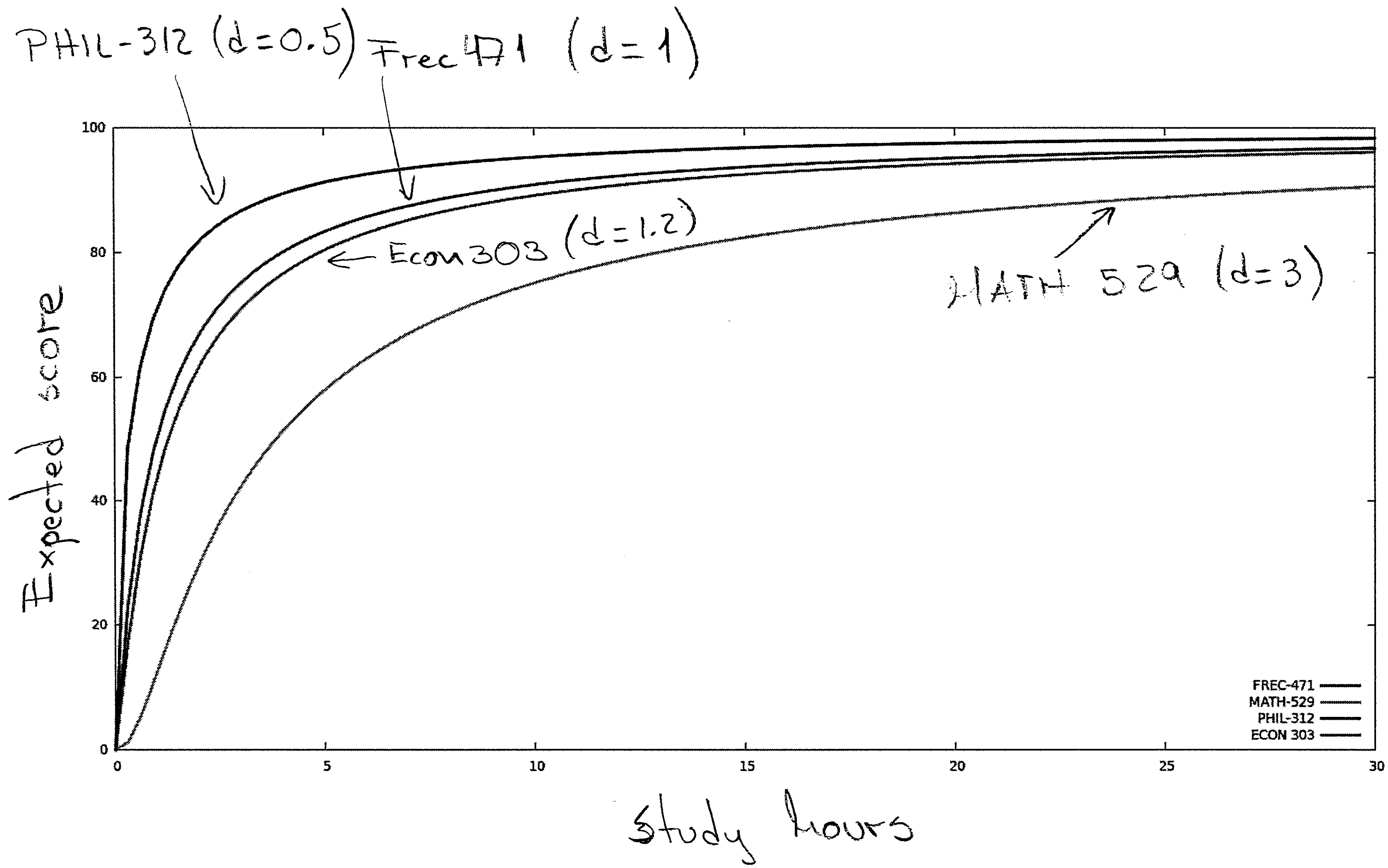
The reduction obtained with the Cauchy step is

from $m_k(0,0) = 1$ to $m_k(-0.2080, -0.1386) = 0.357338$

5. I am going to use the formula

$$g(h,d) = 100\left(\frac{h}{h+1}\right)^d$$

to model the score that Sandrine can obtain after h hours of study on a course with "difficulty index" d. A plot of the behavior of this function is shown:


PHIL-312 (d=0.5)
Frec 471 (d=1)
Econ303 (d=1.2)
MATH 529 (d=3)
100
80
60
40
20
0
0
5
10
15
20
25
30
FREC-471
MATH-529
PHIL-312
ECON 303
Expected score
Study hours

The optimization problem is thus:

$$\max_{h} \underbrace{g(h_1, 1)}_{\text{FREC471}} + \underbrace{g(h_2, 0.5)}_{\text{PHIL 312}} + \underbrace{g(h_3, 1.2)}_{\text{ECON303}} + \underbrace{g(h_4, 3)}_{\text{MATH529}}$$

subject to $h_1 + h_2 + h_3 + h_4 = 30$

Using one of our constrained optimization methods, we may find the following local maximum:

max. value = 341.08 at

$h_1 = 6.702$, $g(h_1, 1) = 87.01$

$h_2 = 7.330$, $g(h_2, 0.5) = 93.80$

$h_3 = 4.714$, $g(h_3, 1.2) = 79.38$

$h_4 = 11.252$, $g(h_4, 3) = 77.45$