MATH529 – Fundamentals of Optimization
Fundamentals of Constrained Optimization

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Motivating Example

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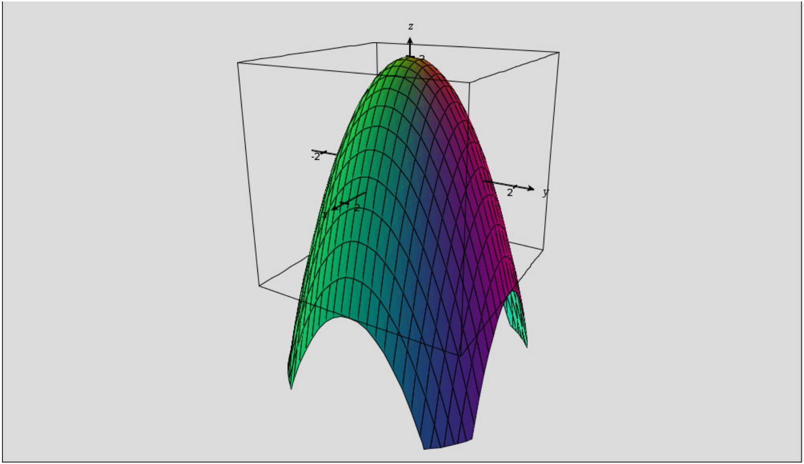
$$\min f(H) = 1000 + 12h_1 + 10h_2 + 11h_3 + \ldots + 15h_n$$

subject to $h_1 + h_2 + \ldots + h_n = 100$, $0 \leq h_i \leq 20$.

Effects of constraints



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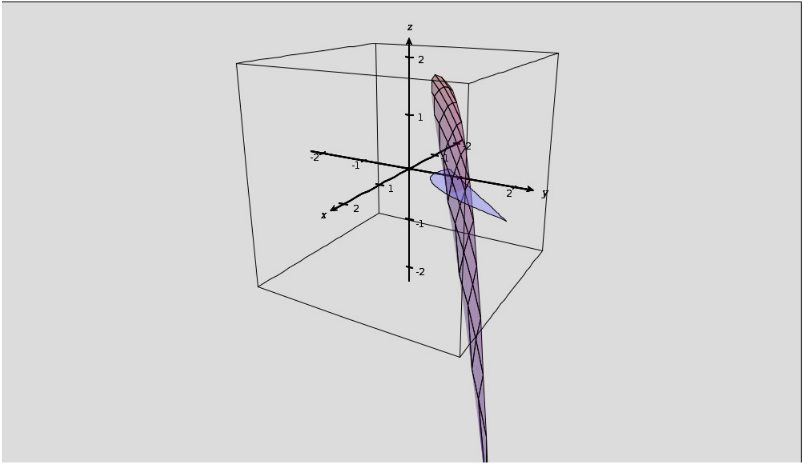


Definitions

A general model of a constrained optimization problem is:

$$\min_{x \in \mathbb{R}^n} f(x)$$

subject to

$$c_i(x) = 0, \quad i \in \mathcal{E}$$

$$c_i(x) \geq 0 \text{ (or } c_i(x) \leq 0), \quad i \in \mathcal{I}$$

where f is called the *objective function*, the functions $c_i(x),\ i \in \mathcal{E}$ are the *equality constraints*, and the functions $c_i(x),\ i \in \mathcal{I}$ are the *inequality constraints*.

Definitions

The feasible set $\Omega \subset \mathbb{R}^n$ is the set of points that satisfy the constraints:

$$\Omega = \{x \mid c_i(x) \geq 0, \ i \in \mathcal{I}, c_i(x) = 0, \ i \in \mathcal{E}\}$$

Therefore, a constrained optimization problem can be defined simply as

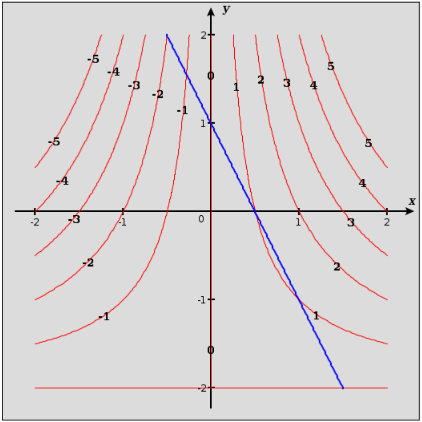
$$\min_{x \in \Omega} f(x)$$

Example: One equality constraint

Suppose you want to solve

$\max f(x) = x_1 x_2 + 2x_1,$

subject to $g(x) = 2x_1 + x_2 = 1.$



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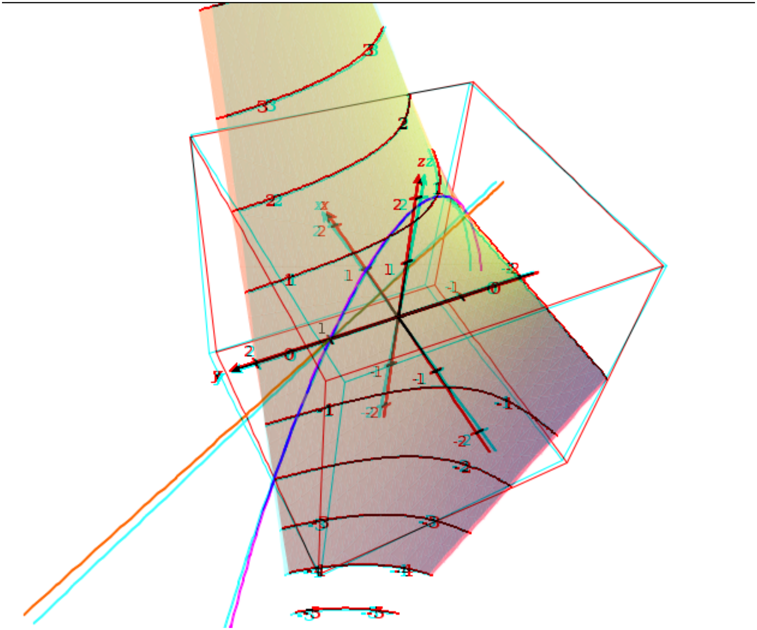

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Solution method 1: Substitution

From the constraint, we see that $x_2 = 1 - 2x_1$. Thus, $f(x)$ can be rewritten as $f(x_1) = x_1(1 - 2x_1) + 2x_1$.

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This method works only on very few cases. E.g., it does not work when one cannot solve for one of the variables.

Example: One equality constraint

Solution method 2: Lagrange multipliers

Define the *Lagrangian function*:

$Z = f(x) + \lambda(0 - c_1(x)) = x_1 x_2 + 2x_1 + \lambda(1 - 2x_1 - x_2)$

where λ is a so-called *Lagrange multiplier*.

If the constraint is satisfied, then $c_1(x) = 0$, and Z is identical to f.

Therefore, as long as $c_1(x) = 0$, searching for the maximum of Z is the same as searching for the maximum of f.

So, **how to always satisfy** $c_1(x) = 0$**?**

Solution method 2: Lagrange multipliers

If $Z = Z(\lambda, x_1, x_2)$, then $\nabla Z = 0$ implies

$\frac{\partial Z}{\partial \lambda} = 1 - 2x_1 - x_2 = 0$, or simply $c_1(x) = 0$ (the original constraint)

$\frac{\partial Z}{\partial x_1} = x_2 + 2 - 2\lambda = 0$, and

$\frac{\partial Z}{\partial x_2} = x_1 - \lambda = 0$

Solving this system: $x_1 = 3/4$, $x_2 = -1/2$, and $\lambda = 3/4$. A second order condition should be used to tell whether $(3/4, -1/2)$ is a maximum or a minimum.

Lagrange multipliers

In general, to find an extremum $x^\star$ of $f(x)$ subject to $c_1(x) = 0$, we define the Lagrangian function:

$$L(x, \lambda) = f(x) - \lambda c_1(x)$$

Then, $\nabla L(x^\star, \lambda^\star) = 0$ implies

$c_1(x^\star) = 0$ and $\nabla f(x^\star) - \lambda^\star \nabla c_1(x^\star) = 0$, or equivalently $\nabla f(x^\star) = \lambda^\star \nabla c_1(x^\star)$.

Exercise

Find all the extrema of $f(x) = x_1^2 + x_2^2$, subject to $x_1^2 + x_2 = 1$

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The first one to finish receives: **an easter bunny!**

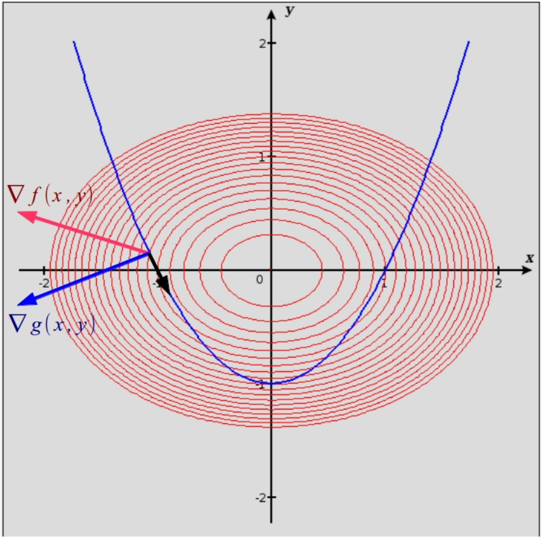
Total differential approach:

Let $z = f(x, y)$ (objective function) and $g(x, y) = c$ (equality constraint). At an extremum, the first order condition translates into $dz = 0$, so $dz = f_x dx + f_y dy = 0$ (1). Since dx and dy are not independent (due to the constraint), we can take the differential of g as well: $dg = g_x dx + g_y dy = 0$ (2). From (2), $dx = -\frac{g_y}{g_x} dy$. Substituting dx in (1):

$-f_x \frac{g_y}{g_x} dy + f_y dy = 0$, which implies $\frac{f_x}{g_x} = \frac{f_y}{g_y}$.

If $\frac{f_x}{g_x} = \frac{f_y}{g_y} = \lambda$, then $f_x - \lambda g_x = 0$ and $f_y - \lambda g_y = 0$, which are the Lagrange multiplier equations.

Taylor series approach:



Lagrange Multiplier Method: Derivation

Taylor series approach:

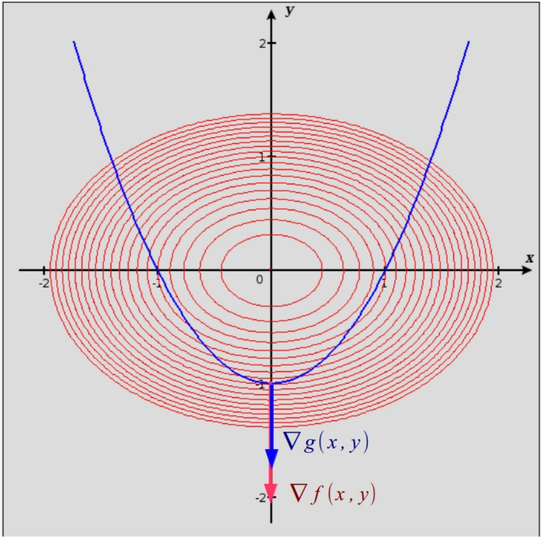
If we want to satisfy the constraint as we move from x to $x + s$, then

$0 = c_1(x + s) \approx c_1(x) + \nabla c_1(x)^T s$, but since $c_1(x) = 0$, then $\nabla c_1(x)^T s = 0$ (1)

Additionally, if we want to decrease the function as we move, we would require $\nabla f(x)^T s < 0$ (2).

Only when $\nabla f(x) = \lambda \nabla g(x)$, we **cannot** find s to satisfy (1) and (2).

Taylor series approach:



Relevance of the Lagrange multiplier in Economics

The Lagrange multiplier measures the sensitivity of the optimal solution to changes in the constraint. For example, assume that $\lambda = \lambda(B)$, $x = x(B)$, and $y = y(B)$. If you want to maximize $U(x)$ subject to $g(x) = B$ (so that $c_1(x) = B - g(x) = 0$). Then,

$L(x, \lambda) = U(x) + \lambda(B - g(x))$. By the Chain Rule:

$$\frac{dL}{dB} = L_{x_1}\frac{dx_1}{dB} + L_{x_2}\frac{dx_2}{dB} + L_\lambda\frac{d\lambda}{dB}$$
$$\frac{dL}{dB} = (U_{x_1} - \lambda g_{x_1})\frac{dx_1}{dB} + (U_{x_2} - \lambda g_{x_2})\frac{dx_2}{dB} + (B - g(x))\frac{d\lambda}{dB} + \lambda(1).$$

Since the first order condition says that $U_{x_1} - \lambda g_{x_1} = 0$, $U_{x_2} - \lambda g_{x_2} = 0$, and $B - g(x) = 0$, we have

$\frac{dL}{dB} = \lambda$. (This equation answers the question "Will a slight relaxation of the budget constraint increase or decrease the optimal value of U?")

Example

Use the Lagrange-multiplier method to find stationary values of $z = x - 3y - xy$, subject to $x + y = 6$. Will a slight relaxation of the constraint increase or decrease the optimal value of z? At what rate?

Multiple constraints

The Lagrangian can be extended to simultaneously consider multiple constraints. For example, let $f(x)$ be subject to:

$g(x) = c$ **and** $h(x) = d$.

The Lagrangian function may be defined as follows:

$L(x, \lambda, \mu) = f(x) + \lambda(c - g(x)) + \mu(d - h(x))$.

The new first-order condition is now:

$c - g(x) = 0,\ d - h(x) = 0,\ f_{x_i} - \lambda g_{x_i} - \mu h_{x_i} = 0.$