MATH529 – Fundamentals of Optimization
Fundamentals of Constrained Optimization III

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Constrained Optimization

A general model of a constrained optimization problem is:

$$\min_{x \in \mathbb{R}^n} f(x)$$

subject to

$$c_i(x) = 0, \quad i \in \mathcal{E}$$

$$c_i(x) \geq 0, \quad i \in \mathcal{I}$$

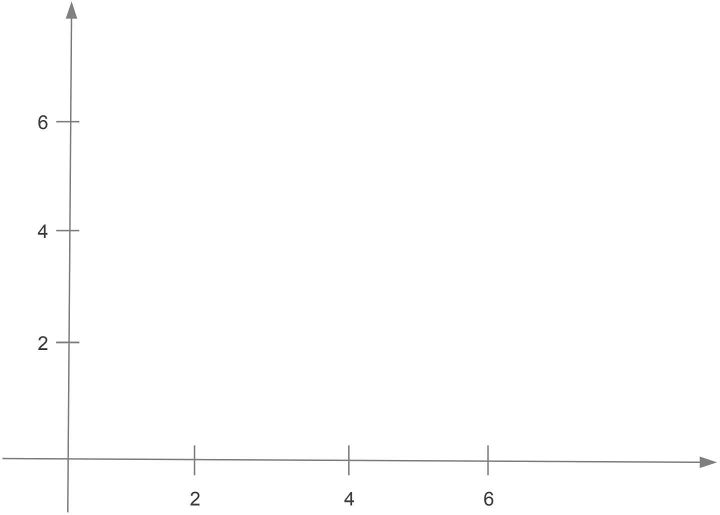
where f is called the *objective function*, the functions $c_i(x), \ i \in \mathcal{E}$ are the *equality constraints*, and the functions $c_i(x), \ i \in \mathcal{I}$ are the *inequality constraints*.

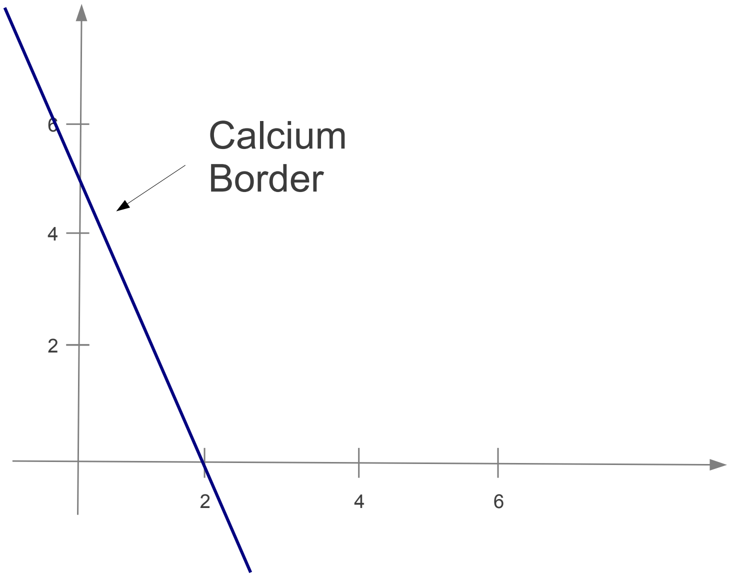
Example: The diet problem

Food 1: $0.6 cts per 100 grms.
Food 2: $1 cts per 100 grms.

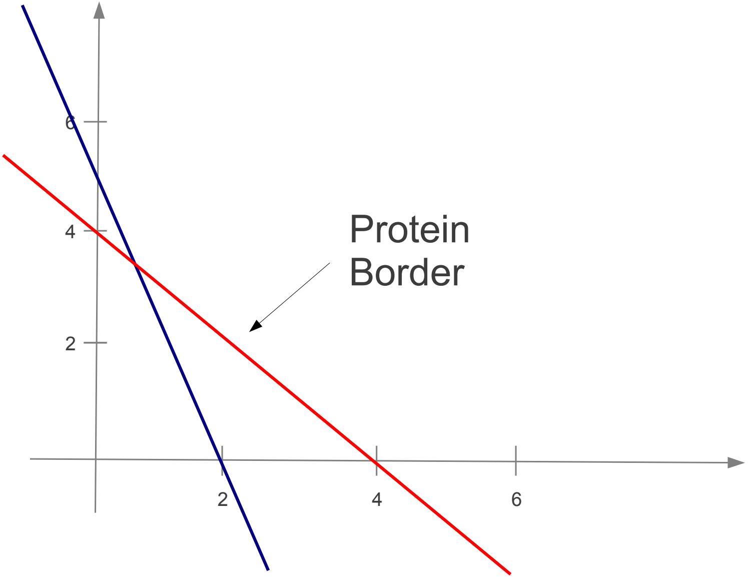
Nutrient	Food 1	Food 2	Minimum Daily Requirement
Calcium	10	4	20
Protein	5	5	20
Vitamins	2	6	12

Graphical Solution

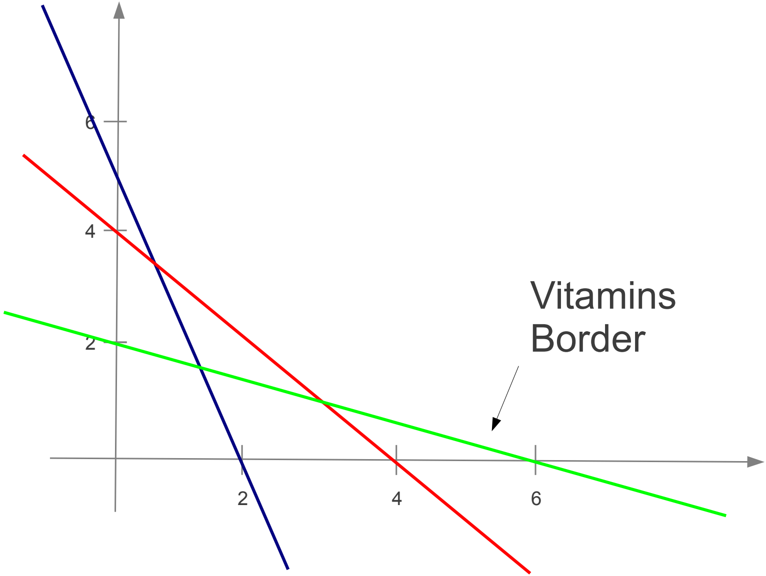



Calcium
Border
6
4
2
2
4
6

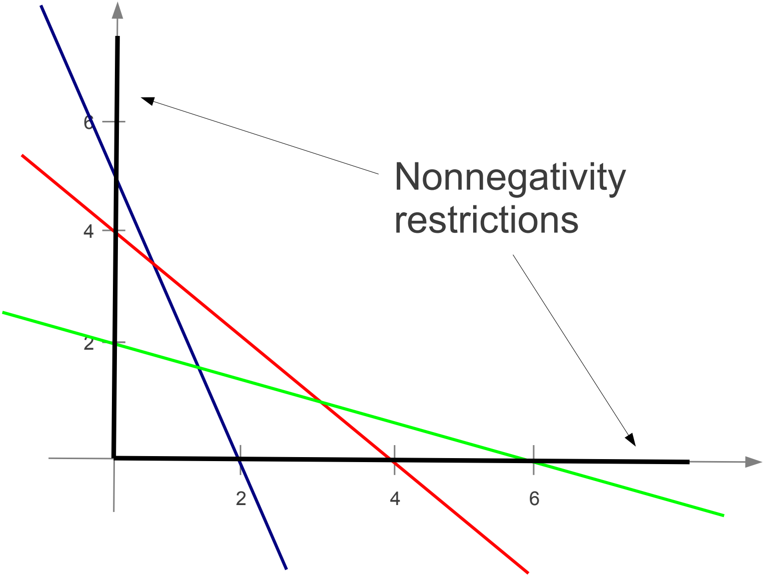
Graphical Solution



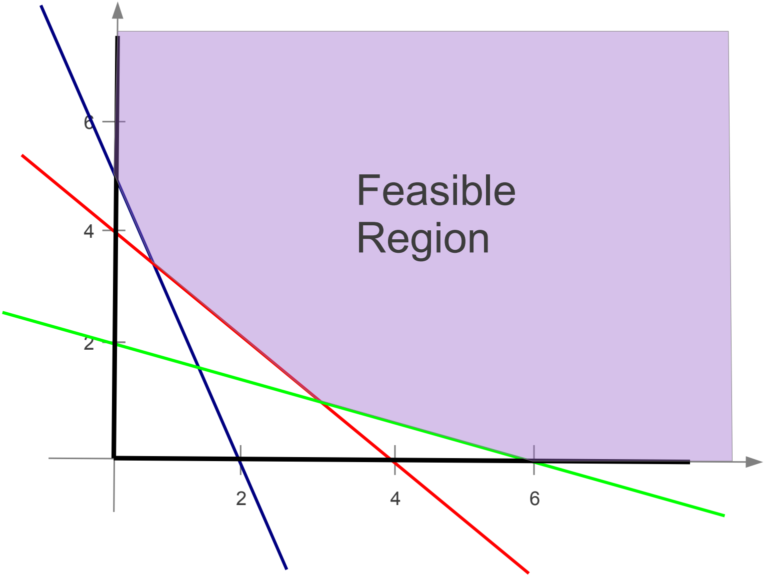
Graphical Solution



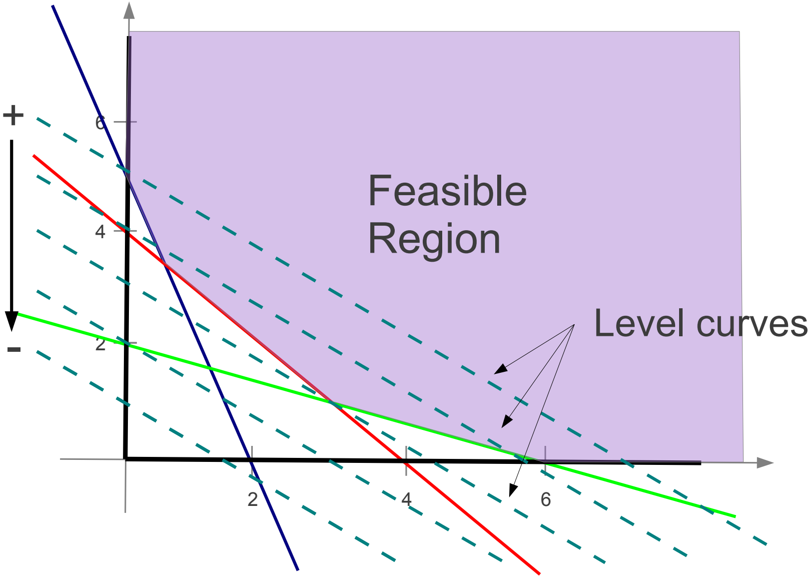
Graphical Solution



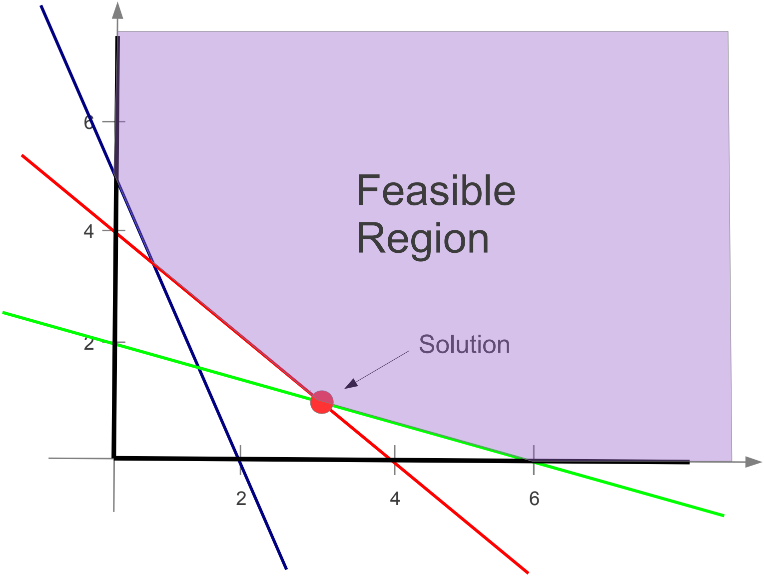
Graphical Solution



Graphical Solution



Graphical Solution



Equality vs. Inequality Constraints

Equality

$$\text{minimize} \quad x_1 + x_2$$

$$10x_1 + 4x_2 = 20$$

$$5x_1 + 5x_2 = 20$$

$$2x_1 + 12x_2 = 12$$

Inequality

$$\text{minimize} \quad x_1 + x_2$$

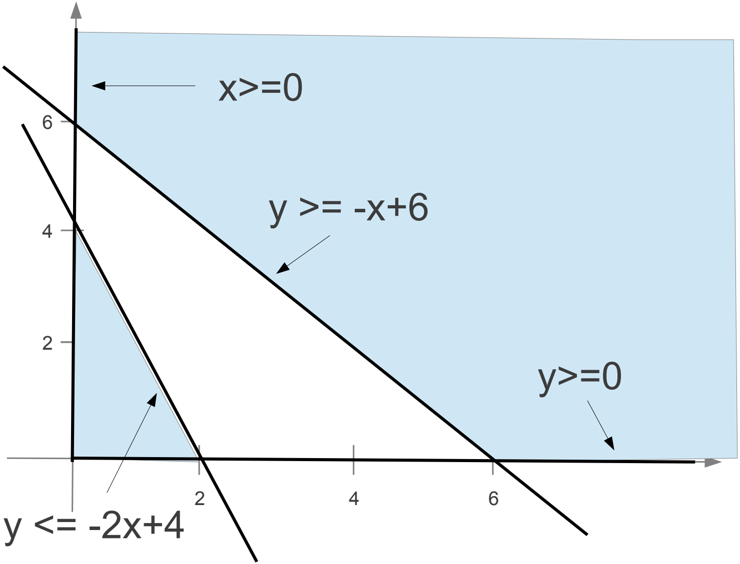
$$10x_1 + 4x_2 \geq 20$$

$$5x_1 + 5x_2 \geq 20$$

$$2x_1 + 12x_2 \geq 12$$

$$x_1,\ x_2 \geq 0$$

What can go wrong?



Active Set and Constraints

Definition

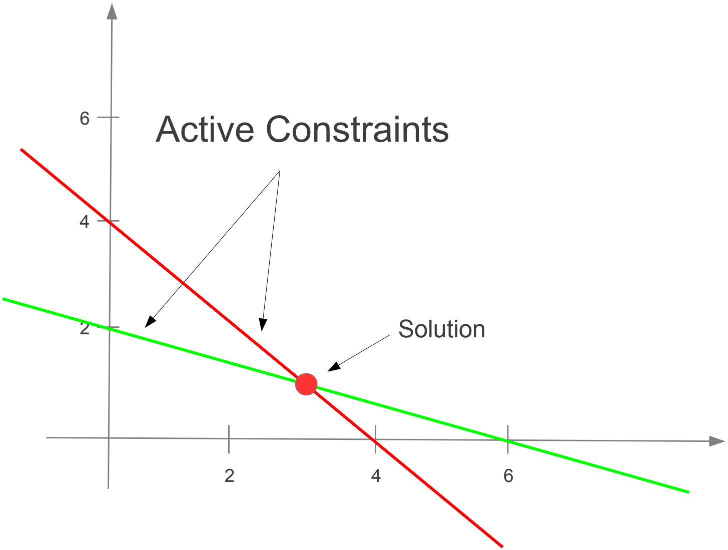
An *active* inequality constraint i at a point x is one for which $c_i(x) = 0$. An *inactive* inequality constraint j at x is one for which $c_j(x) > 0$.

Definition

The active set $\mathcal{A}(x)$ at any feasible point x is the set of indices of all the equality constraints together with the indices i of the active inequality constraints.

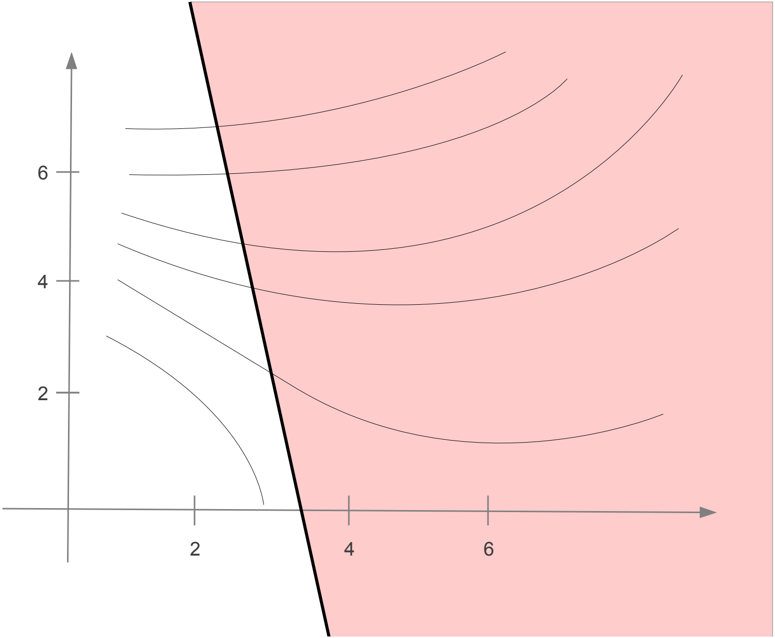
In our example

At the solution $x = (3, 1)^T$, $\mathcal{A}(x) = \{2, 3\}$.

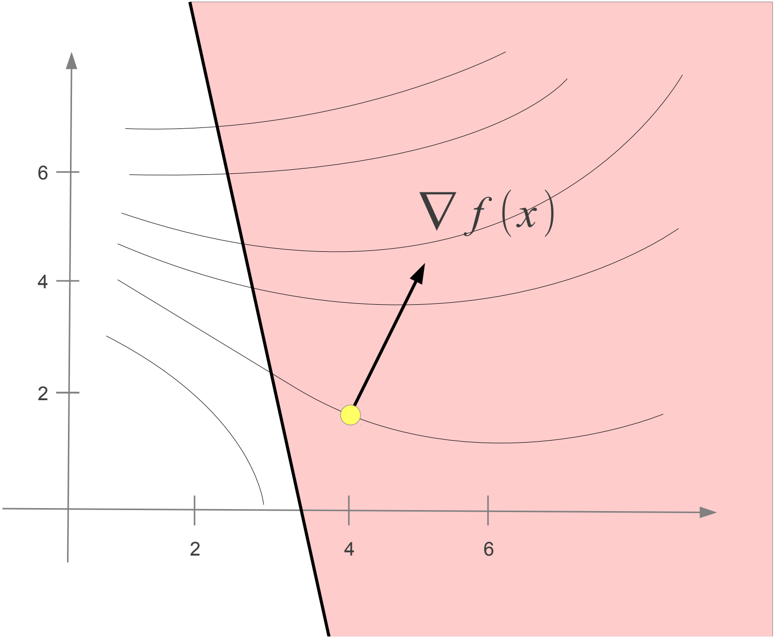


Karush-Kuhn-Tucker (KKT) Conditions

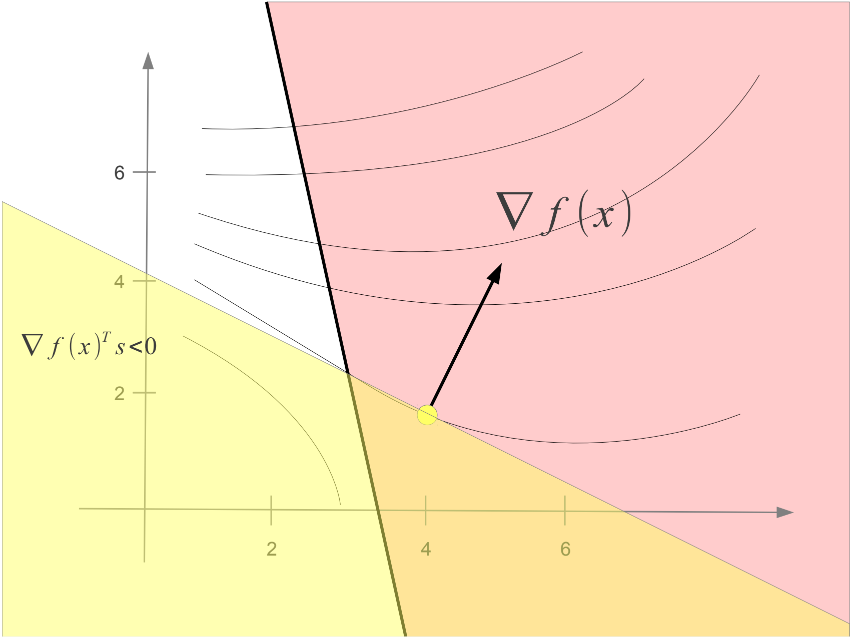
Decrease Conditions: Interior Point



Decrease Conditions: Interior Point



Decrease Conditions: Interior Point



Decrease Conditions: Interior Points

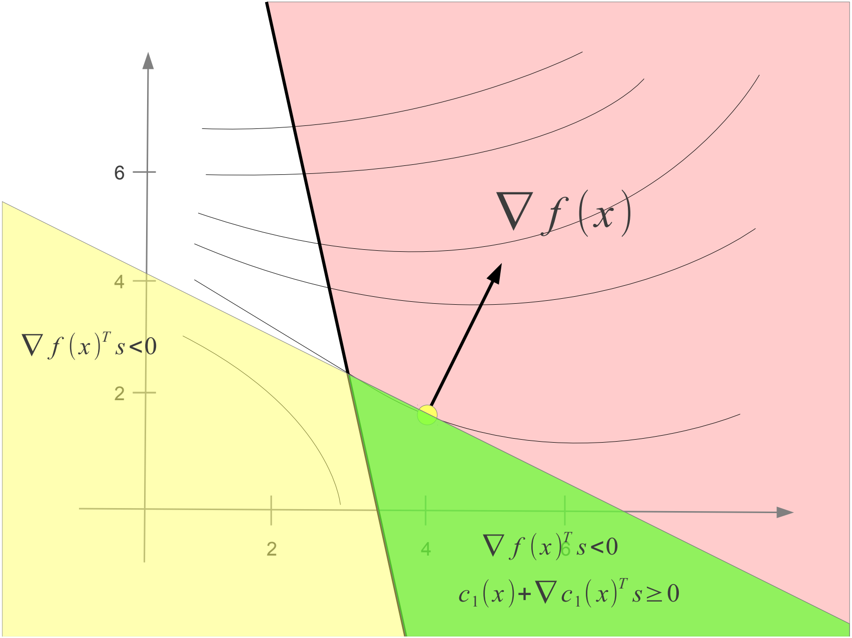
At a point x, a step s that would result in an objective function should satisfy:

$\nabla f(x)^T s < 0$ (objective function decrease)*

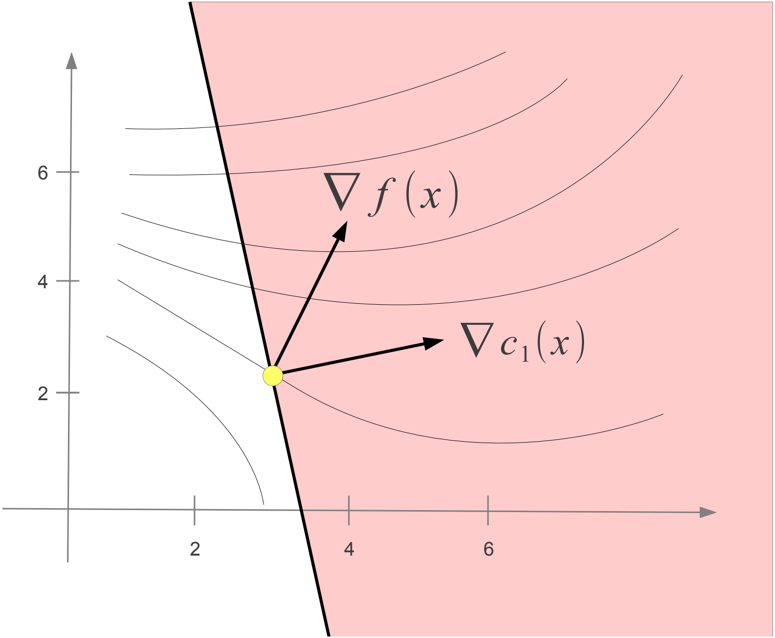
$0 \leq c(x+s) \approx c(x) + \nabla c(x)^T s$ (feasibility)

* Unless $\nabla f(x) = 0$

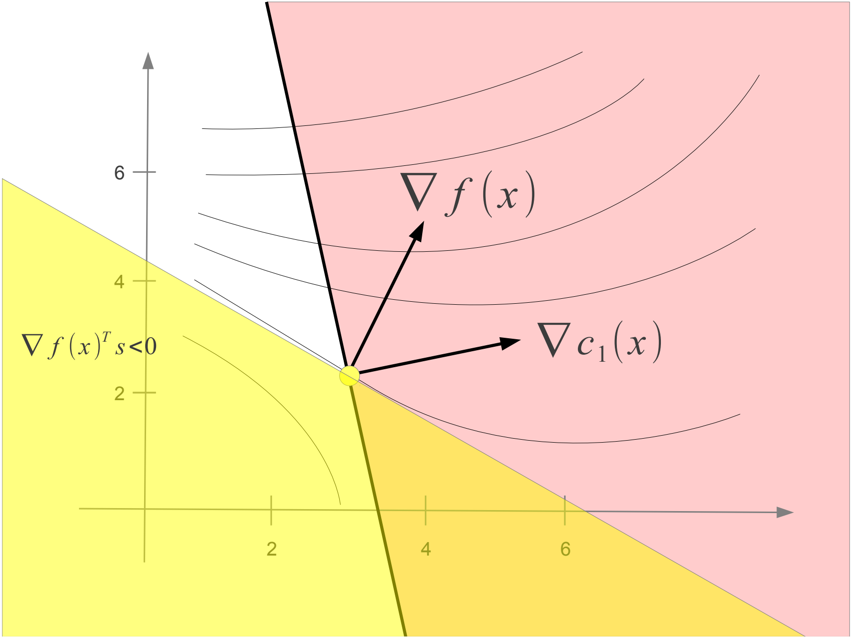
Decrease Conditions: Border Point



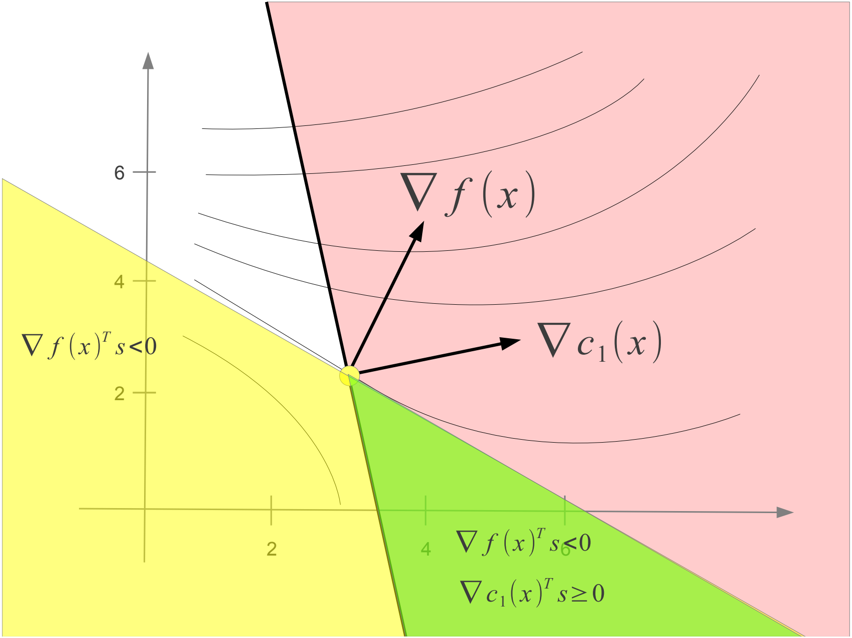
Decrease Conditions: Border Point



Decrease Conditions: Border Point



Conditions: Border Point



Decrease Conditions: Border Points

At a point x, a step s that would result in an objective function should satisfy:

$\nabla f(x)^T s < 0$ (objective function decrease)

$0 \leq c(x+s) \approx c(x) + \nabla c(x)^T s$ (feasibility)

However, since x is a border point, $c(x) = 0$,:

$0 \leq \nabla c(x)^T s$ (feasibility)

Conditions: General

When no step s can be found, the point $x^\star$ is a candidate for a solution. The conditions for interior and border cases, can be summarized as follows:

$\nabla_x L(x^\star, \lambda^\star) = 0,$
$c_i(x) = 0, \quad \text{for all } i \in \mathcal{E}$
$c_i(x) \geq 0, \quad \text{for all } i \in \mathcal{I}$
$\lambda^\star c_i(x^\star) = 0, \quad \text{for all } i \in \mathcal{E} \cup \mathcal{I}$ (complementarity condition), and
$\lambda^\star \geq 0.$

Example

$$\text{minimize } x^2 + 2y^2$$

subject to:

$$x + y \geq 0$$
$$y - x^2 \geq 1$$

Example

maximize $x + y^2$

subject to:

$$x - y = 5$$
$$x^2 + 9y^2 \leq 36$$