MATH529 – Fundamentals of Optimization
Fundamentals of Constrained Optimization IV

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Example

maximize $x + y^2$

subject to:

$$x - y = 5$$
$$x^2 + 9y^2 \leq 25$$

Constraint Qualifications: Motivating Examples

Example:

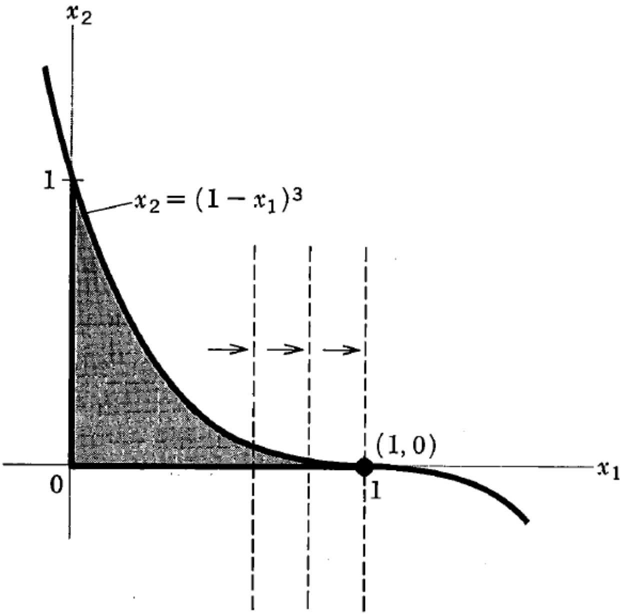
$$\text{maximize } x_1$$

$$\text{subject to:}$$

$$x_2 - (1 - x_1)^3 \leq 0$$

$$x_1, x_2 \geq 0$$

Constraint Qualifications: Motivating Examples



Constraint Qualifications: Motivating Examples

Example:

$$\text{maximize } x_1$$

subject to:

$$x_2 - (1 - x_1)^3 \leq 0$$

$$2x_1 + x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Constraint Qualifications: Motivating Examples

$$L(x,\lambda) = x_1 + \lambda_1(-x_2 + (1-x_1)^3) + \lambda_2(2-2x_1-x_2) + \lambda_3 x_1 + \lambda_4 x_2$$

KKT conditions:

(1) $1 - 3\lambda_1(1-x_1)^2 - 2\lambda_2 + \lambda_3 = 0$

(2) $-\lambda_1 - \lambda_2 + \lambda_4 = 0$

(3) $x_2 - (1-x_1)^3 \leq 0$

(4) $2x_1 + x_2 \leq 2$

(5) $x_1, x_2 \geq 0$

(6) $\lambda_1(-x_2 + (1-x_1)^3) = 0$, $\lambda_2(2-2x_1-x_2) = 0$, $\lambda_3 x_1 = 0$, $\lambda_4 x_2 = 0$

(7) $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$

Constraint Qualifications: Motivating Examples

At $(1, 0)$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 = 0$, $\lambda_4 \geq 0$. Then:

(1) $1 - 2\lambda_2 = 0$, which implies $\lambda_2 = \frac{1}{2}$

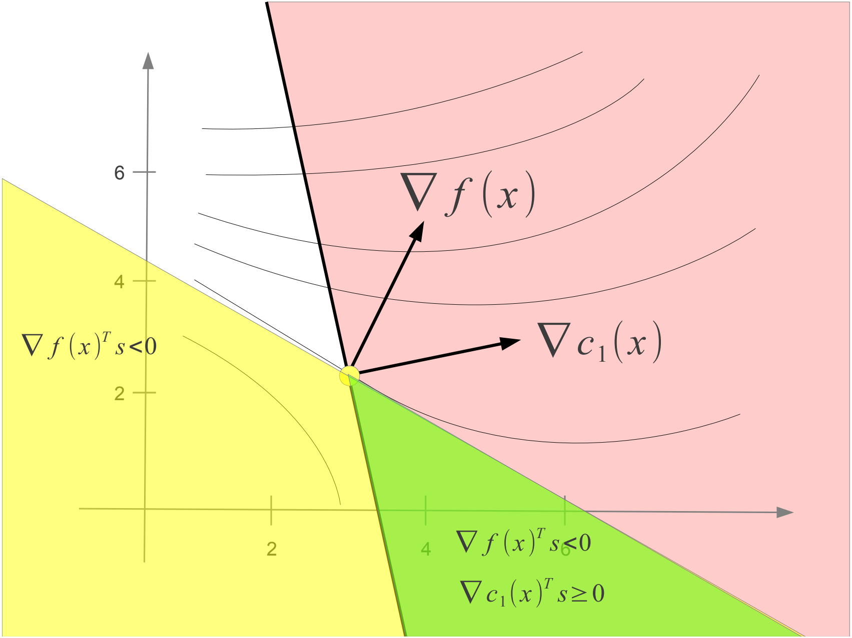
(2) $-\lambda_1 - \lambda_2 + \lambda_4 = -\lambda_1 - \frac{1}{2} + \lambda_4 = 0$, or $-\lambda_1 + \lambda_4 = \frac{1}{2}$

Thus, $(1, 0)$ satisfies the KKT conditions as long as $-\lambda_1 + \lambda_4 = \frac{1}{2}$ for $\lambda_1, \lambda_2 \geq 0$.

a) **The vector of Lagrange multipliers is not necessarily unique.**
b) **KKT conditions can remain valid despite the existence of cusps.**
c) **There are cases in which the KKT conditions fail even without cusps.**

Why?

Constraint Qualifications

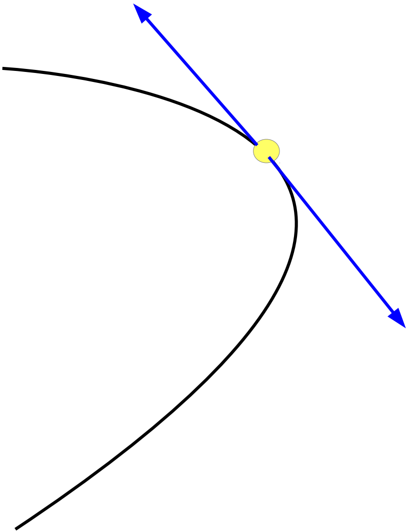


Constraint Qualifications: Tangent Cone

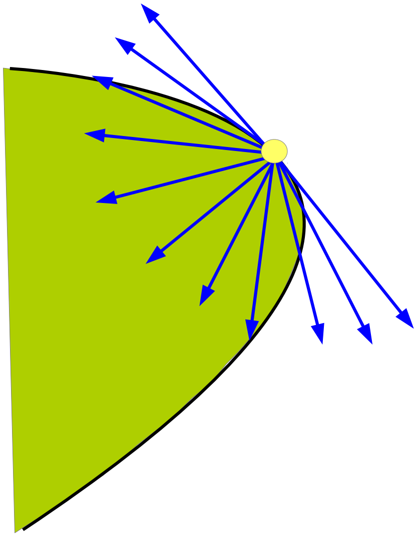
Definition

The tangent cone to a set Ω at a point $\mathbf{x} \in \Omega$, denoted by $T_\Omega(\mathbf{x})$, consists of the limits of all (secant) rays which originate at $\mathbf{x}$ and pass through a sequence of points $\mathbf{p}_i \in \Omega - \{\mathbf{x}\}$ which converges to $\mathbf{x}$.

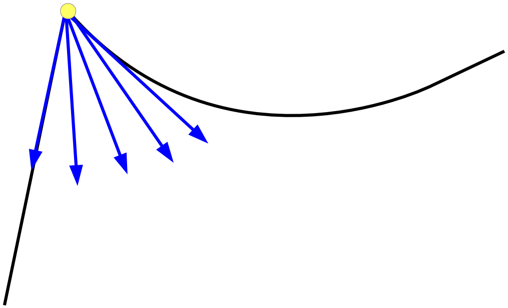
Constraint Qualifications: Tangent Cone



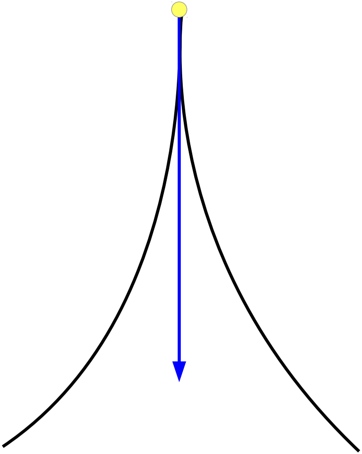
Constraint Qualifications: Tangent Cone



Constraint Qualifications: Tangent Cone



Constraint Qualifications: Tangent Cone



Constraint Qualifications: Linearized feasible directions set

Definition

Given a feasible point $\mathbf{x}$ and the active constraint set $\mathcal{A}(\mathbf{x})$, the set of linearized feasible directions $\mathcal{F}(\mathbf{x})$ is the set of vectors $\mathbf{d}$ such that

$$\begin{cases} \mathbf{d}^T \nabla c_i(\mathbf{x}) = 0 & \text{for all } i \in \mathcal{E}, \\ \mathbf{d}^T \nabla c_i(\mathbf{x}) \geq 0 & \text{for all } i \in \mathcal{A}(\mathbf{x}) \bigcap \mathcal{I}. \end{cases}$$

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The definition of $T_\Omega(\mathbf{x})$ depends on the geometry of Ω. The definition of $\mathcal{F}(\mathbf{x})$ depends on the algebraic definition of the constraints.

Constraint Qualifications: Motivating Examples

maximize x_1

subject to:

$$x_2 - (1 - x_1)^3 \leq 0$$

$$x_1, x_2 \geq 0$$

maximize x_1

subject to:

$$x_2 - (1 - x_1)^3 \leq 0$$

$$2x_1 + x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Constraint Qualifications: Definition

Definition

A constraint qualification is an assumption that ensures similarity of the constraint set Ω and its linearized approximation, in a neighborhood of a point $\mathbf{x}^\star$.

Constraint qualifications are *sufficient* conditions for the linear approximation to be adequate. However, they are not necessary.

Constraint Qualifications: LICQ

Definition

Given a point $\mathbf{x}$ and the active set $\mathcal{A}(\mathbf{x})$, we say that the linear independence constraint qualification (LICQ) holds if the set of active constraint gradients $\{\nabla c_i(\mathbf{x}), i \in \mathcal{A}(\mathbf{x})\}$ is linearly independent.

Constraint Qualifications: LICQ

Example:

$$\text{maximize } x_1$$

$$\text{subject to:}$$

$$x_2 - (1 - x_1)^3 \leq 0$$

$$x_1, x_2 \geq 0$$

At $\mathbf{x} = (1, 0)$, $\mathcal{A}(\mathbf{x}) = \{1, 3\}$.

$c_1(\mathbf{x}) = x_2 - (1 - x_1)^3$, so $\nabla c_1(1, 0) = (0, 1)^T$

$c_3(\mathbf{x}) = -x_2$, so $\nabla c_3(1, 0) = (0, -1)^T$.

Clearly, $\nabla c_1(1, 0)$ and $\nabla c_3(1, 0)$ are not linearly independent.

Example

$$\text{maximize } x_1$$

subject to:

$$x_1^2 + x_2^2 \leq 1$$

$$x_1, x_2 \geq 0$$

Solve graphically, draw the tangent cone and the set of feasible directions at the solution point, check also whether the optimal point satisfies LICQ, and the KKT conditions.

Constraint Qualifications: LICQ

Some implications:

- In general, there may be many vectors $\lambda^\star$ that satisfy the KKT conditions at a solution point $\mathbf{x}^\star$. However, if LICQ holds, then $\lambda^\star$ is unique.
- If all the active constraints are linear, then $\mathcal{F}(\mathbf{x}^\star) = T_\Omega(\mathbf{x}^\star)$.

Constraint Qualifications: MFCQ

Another constraint qualification is called **Mangasarian-Fromovitz**.

Definition

Given a point $\mathbf{x}$ and the active set $\mathcal{A}(\mathbf{x})$, we say that the Mangasarian-Fromovitz (MFCQ) holds if there exists a vector $\mathbf{w} \in \mathbb{R}^n$ such that

$$\nabla c_i(\mathbf{x}^\star)^T \mathbf{w} > 0 \text{ for all } i \in \mathcal{A}(\mathbf{x}) \cap \mathcal{I}$$

$$\nabla c_i(\mathbf{x}^\star)^T \mathbf{w} = 0, \text{ for all } i \in \mathcal{E}$$

Relationship between LICQ and MFCQ

If $\mathbf{x}^\star \in \Omega$ satisfies LICQ, then $\mathbf{x}^\star$ satisfies MFCQ.

Proof: Suppose we are minimizing a function $f(\mathbf{x})$. Define $\mathcal{A}(\mathbf{x}^\star) = \{1, 2, \ldots, m, m+1, \ldots, q\}$ where $1, 2, \ldots, m$ are the indices of all the equality constraints, and $m+1, \ldots, q$ are the indices of all the active inequality constraints. Then define

$$M = \begin{pmatrix} \nabla c_1(\mathbf{x}^\star) \\ \vdots \\ \nabla c_m(\mathbf{x}^\star) \\ \nabla c_{m+1}(\mathbf{x}^\star) \\ \vdots \\ \nabla c_q(\mathbf{x}^\star) \end{pmatrix}$$

By LICQ, the rows of M are linearly independent.

Relationship between LICQ and MFCQ

Therefore, the system $M\mathbf{d} = \mathbf{b}$ should have a solution, for some $\mathbf{d} \in \mathbb{R}^q$ and $\mathbf{b} = (0, 0, \ldots, 0, 1, \ldots, 1)^T$. (The first m terms are all zero, and the rest all one.)

The solution vector $\mathbf{d}$ ensures that $\nabla c_i(\mathbf{x}^\star)^T\mathbf{d} = 0$, for all $i \in \mathcal{E}$, and $\nabla c_j(\mathbf{x}^\star)^T\mathbf{d} = 1 > 0$, for all $j \in \mathcal{A}(\mathbf{x}^\star) \cap \mathcal{I}$.

Relationship between LICQ and MFCQ

MFCQ does not imply LICQ.

Example: Check $\mathbf{x}^\star = (0,0)^T$ for:

max $f(x,y)$ subject to

$$(x-1)^2 + (y-1)^2 \le 2$$
$$(x-1)^2 + (y+1)^2 \le 2$$
$$-x \le 0$$