MATH529 – Fundamentals of Optimization

Fundamentals of Constrained Optimization V: Linear Programming and the Simplex Method

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Methods for Constrained Optimization

Optimization Algorithms exploit the structure of the problem:

- Linear Programs: Simplex Method, Interior-point Method, ...
- Nonlinear Programs: Sequential Quadratic Programming, Penalty Methods, ...

Linear Programming

Definition

A linear program is one with linear objective function and linear constraints, which may include equality and inequality constraints.

Example:

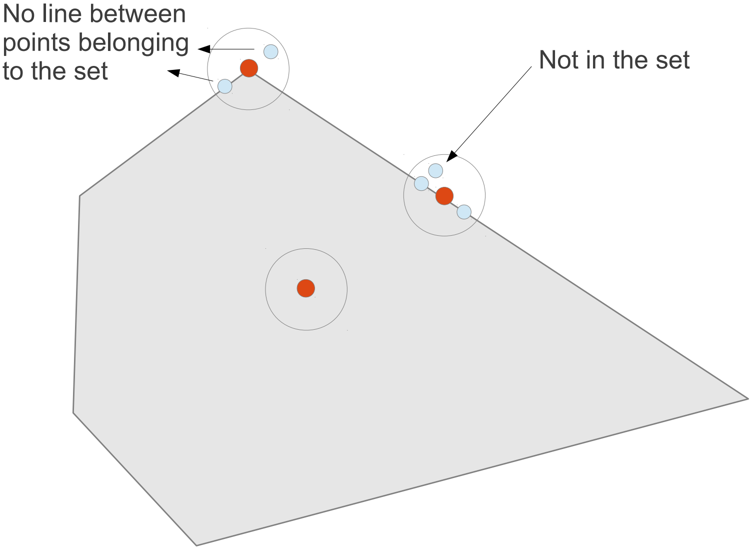
$$\min \ -4x_1 + 2x_2 - 2x_3 + 13x_4,$$

$$\text{subject to } x_1 + x_2 \geq 3, \ x_1 - x_3 + x_4 = 3, \ x_1, x_2, x_3, x_4 \geq 0$$

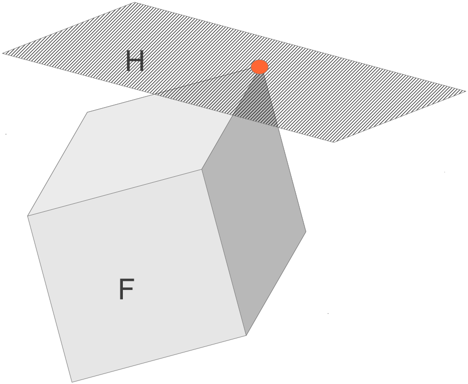
Appeal:

- It is often easier (and more appropriate) to model problems as linear programs
- KKT conditions are valid
- The feasible set is a *polytope* (a convex, connected, set with flat, polygonal faces).
- The solution is found at extreme points, thus reducing the search space to specific regions of the feasible set.
- A local optimum will also be a global optimum.

Interior, boundary, extreme points:



Supporting hyperplanes:



A supporting hyperplane H has one or more points in common with a convex set F, but F lies completely one one side of H.

- Theorem 1: Given $\mathbf{u}$, a boundary point of a closed convex set, there is at least one supporting hyperplane at $\mathbf{u}$.
- Theorem 2: For a closed convex set bounded from below, there is at least one extreme point in every supporting hyperplane.

Since in linear programming the objective function is linear, the hyperplane corresponding to the optimal value of the function will be a supporting hyperplane of the feasible set.

- Theorem 1: Given $\mathbf{u}$, a boundary point of a closed convex set, there is at least one supporting hyperplane at $\mathbf{u}$.
- Theorem 2: For a closed convex set bounded from below, there is at least one extreme point in every supporting hyperplane.

Since in linear programming the objective function is linear, the hyperplane corresponding to the optimal value of the function will be a supporting hyperplane of the feasible set.

We can therefore pay attention only to extreme points!

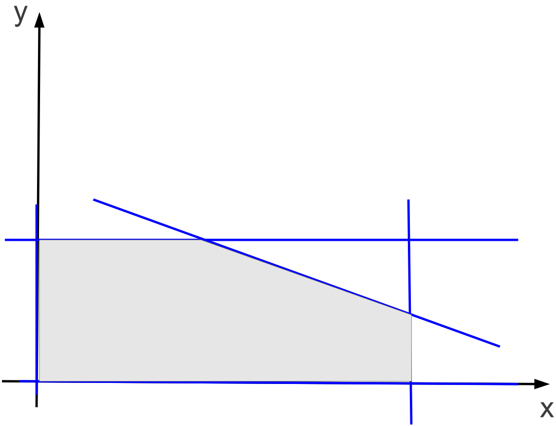
Linear Programming: Toward a solution method

Example:

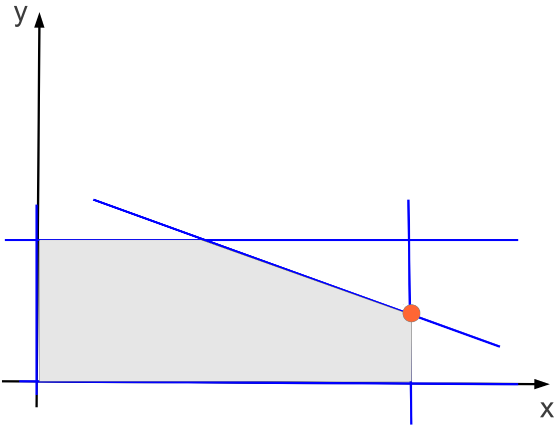
$\max\ 40x + 30y$

subject to:

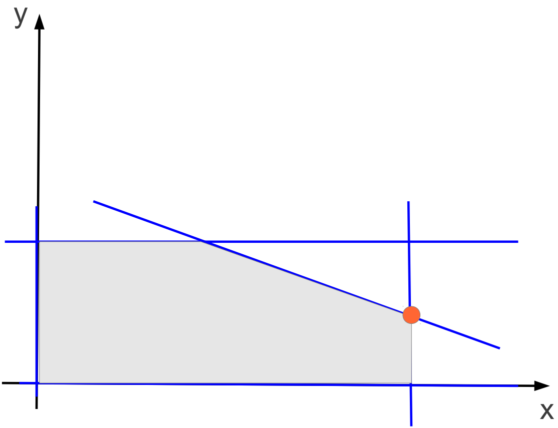
$$x + 2y \leq 24$$
$$0 \leq x \leq 16$$
$$0 \leq y \leq 8$$


y
x

Dummy variables: Slacks and Surpluses:



Dummy variables: Slacks and Surpluses:



With the inexact satisfaction of a constraint, there is a slack or surplus related to one or more constraints.

Step 1: Transforming the program:

max $p = 40x + 30y + 0s_1 + 0s_2 + 0s_3 + 0s_4 + 0s_5$

subject to:

$x + 2y + s_1 = 24$
$x + s_2 = 16$
$y + s_3 = 8$
$x - s_4 = 0\,^*$
$y - s_5 = 0\,^*$

$x, y, s_1, s_2, s_3, s_4, s_5 \geq 0$

*For surpluses, we add $-s_i$, with $s_i \geq 0$.

Linear Programming: Toward a solution method

Step 1: Transforming the program:

max $p = 40x + 30y$

subject to:

$x + 2y + s_1 = 24$
$x + s_2 = 16$
$y + s_3 = 8$

$x, y, s_1, s_2, s_3 \geq 0$

Linear Programming: Toward a solution method

Step 2: Generating extreme points:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

Step 2: Generating extreme points:

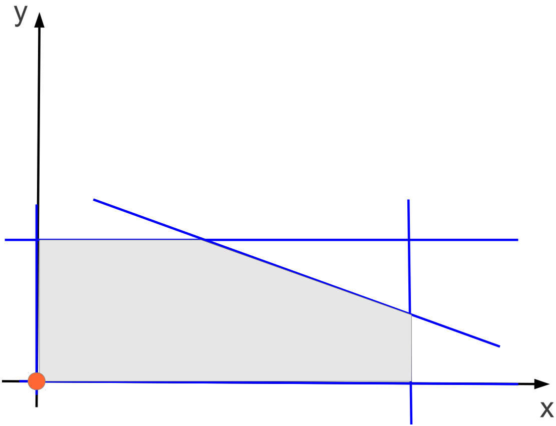
$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

Let $x = y = 0$, then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

That is $s_1 = 24$, $s_2 = 16$, and $s_3 = 8$.

Linear Programming: Toward a solution method



Step 2: Generating extreme points:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

Let $x = s_1 = 0$, then

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

That is $y = 12$, $s_2 = 16$, and $s_3 = -4$. (Violates nonnegativity restrictions!)

Step 2: Generating extreme points:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

Let $x = s_2 = 0$, then

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ s_1 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

Invalid system!

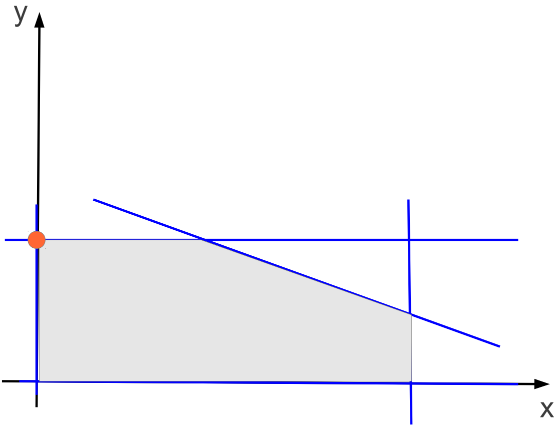
Step 2: Generating extreme points:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

Let $x = s_3 = 0$, then

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

That is $y = 8$, $s_2 = 16$, and $s_3 = 8$.


y
x

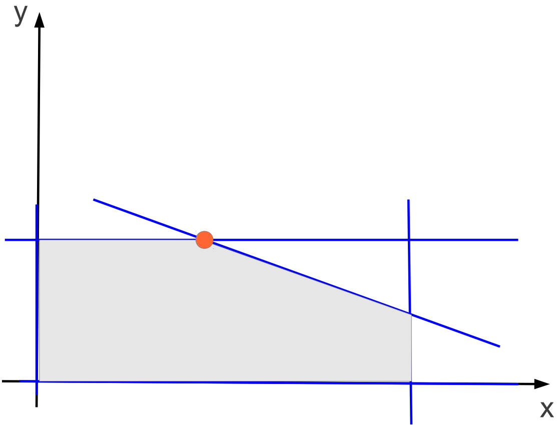
Step 2: Generating extreme points:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

Let $s_1 = s_3 = 0$, then

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_2 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

That is $x = 8$, $y = 8$, and $s_2 = 8$.


y
x

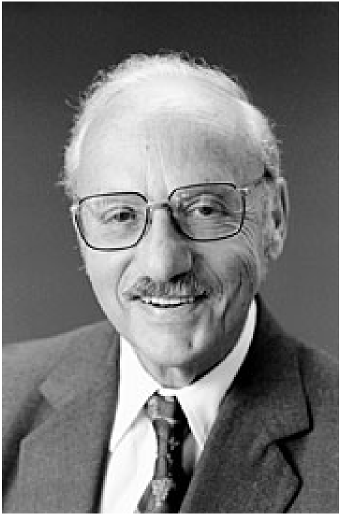
Linear Programming: Toward a solution method

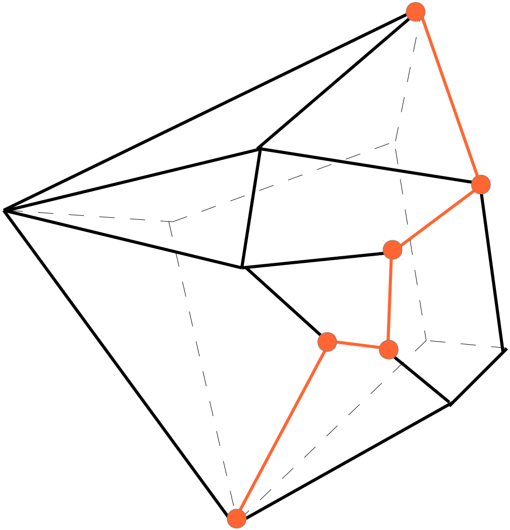
We now have a method to systematically generate extreme points. But there are a few remaining questions:

- How to select which columns to eliminate so that we generate as few extreme points as possible?
- How can we use the objective function to guide the search?
- How to avoid generating invalid moves?

The Simplex Method

George B. Dantzig





The Simplex Method

Simplex tableau:

p	x	y	s_1	s_2	s_3	Constant
1	-40	-30	0	0	0	0
0	1	2	1	0	0	24
0	1	0	0	1	0	16
0	0	1	0	0	1	8

Basis at (0,0):

$$\begin{bmatrix} s_1 & s_2 & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pivoting: Choose the pivot column, then the pivot element.

p	x	y	s_1	s_2	s_3	Constant
1	-40	-30	0	0	0	0
0	1	2	1	0	0	24
0	1	0	0	1	0	16
0	0	1	0	0	1	8

Pivoting: Choose the pivot column, then the pivot element.

Choose the column associated with the negative entry with the largest abs. value

p	x	y	s_1	s_2	s_3	Constant
1	**-40**	-30	0	0	0	0
0	**1**	2	1	0	0	24
0	**1**	0	0	1	0	16
0	**0**	1	0	0	1	8

The Simplex Method

Pivoting: Choose the pivot column, then the pivot element.

If we choose to replace s_1:

p	x	y	s_1	s_2	s_3	Constant	
1	-40	-30	0	0	0	0	$40R_2 + R_1$
0	**1**	2	1	0	0	24	
0	1	0	0	1	0	16	$R_2 - R_3$
0	0	1	0	0	1	8	

Pivoting: Choose the pivot column, then the pivot element.

If we choose to replace s_1:

p	x	y	s_1	s_2	s_3	Constant
1	0	50	40	0	0	960
0	**1**	2	1	0	0	24
0	0	2	1	-1	0	8
0	0	1	0	0	1	8

New basis:

$$\begin{bmatrix} x & s_2 & s_3 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $x = 24$, $y = 0$, $s_1 = 0$, $s_2 = -8$, $s_3 = 8$

Pivoting: Choose the pivot column, then the pivot element.

If we choose to replace s_1:

p	x	y	s_1	s_2	s_3	Constant
1	0	50	40	0	0	960
0	**1**	2	1	0	0	24
0	0	2	1	-1	0	8
0	0	1	0	0	1	8

New basis:

$$\begin{bmatrix} x & s_2 & s_3 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $x = 24,\ y = 0,\ s_1 = 0,\ s_2 = -8,\ s_3 = 8$

Invalid move!

Pivoting: Choose the pivot column, then the pivot element.

If we choose to replace s_2:

p	x	y	s_1	s_2	s_3	Constant	
1	-40	-30	0	0	0	0	$40R_3 + R_1$
0	1	2	1	0	0	24	$R_2 - R_3$
0	**1**	0	0	1	0	16	
0	0	1	0	0	1	8	

Pivoting: Choose the pivot column, then the pivot element.

If we choose to replace s_2:

p	x	y	s_1	s_2	s_3	Constant
1	0	-30	0	40	0	640
0	0	2	1	-1	0	8
0	**1**	0	0	1	0	16
0	0	1	0	0	1	8

New basis:

$$\begin{bmatrix} s_1 & x & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $x = 16$, $y = 0$, $s_1 = 8$, $s_2 = 0$, $s_3 = 8$

Pivoting: Choose the pivot column, then the pivot element.

So how do we choose the pivot element?

Pick the positive elements in the pivot column,
Divide the constant column by these elements,
Select the element corresponding to the smallest quotient as pivot element

Pivoting: Choose the pivot column, then the pivot element.

p	x	y	s_1	s_2	s_3	Constant	
1	-40	-30	0	0	0	0	
0	1	2	1	0	0	24	24/1
0	**1**	0	0	1	0	16	16/1
0	0	1	0	0	1	8	

The Simplex Method

A second move:

p	x	y	s_1	s_2	s_3	Constant
1	0	-30	0	40	0	640
0	0	2	1	-1	0	8
0	1	0	0	1	0	16
0	0	1	0	0	1	8

A second move:

p	x	y	s_1	s_2	s_3	Constant
1	0	**-30**	0	40	0	640
0	0	**2**	1	-1	0	8
0	1	**0**	0	1	0	16
0	0	**1**	0	0	1	8

A second move:

p	x	y	s_1	s_2	s_3	Constant	
1	0	-30	0	40	0	640	$30R_2/2 + R_1$
0	0	**2**	1	-1	0	8	$R_2/2$
0	1	0	0	1	0	16	
0	0	1	0	0	1	8	$R_4 - R_2/2$

A second move:

p	x	y	s_1	s_2	s_3	Constant
1	0	0	15	25	0	760
0	0	1	1/2	-1/2	0	4
0	1	0	0	1	0	16
0	0	0	-1/2	1/2	1	4

New basis:

$$\begin{bmatrix} y & x & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $x = 16,\ y = 4,\ s_1 = 0,\ s_2 = 0,\ s_3 = 8$

A second move:

p	x	y	s_1	s_2	s_3	Constant
1	0	0	15	25	0	760
0	0	1	1/2	-1/2	0	4
0	1	0	0	1	0	16
0	0	0	-1/2	1/2	1	4

New basis:

$$\begin{bmatrix} y & x & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $x = 16,\ y = 4,\ s_1 = 0,\ s_2 = 0,\ s_3 = 8$

Since there are no more negative entries in the first row, we have reached the optimal solution

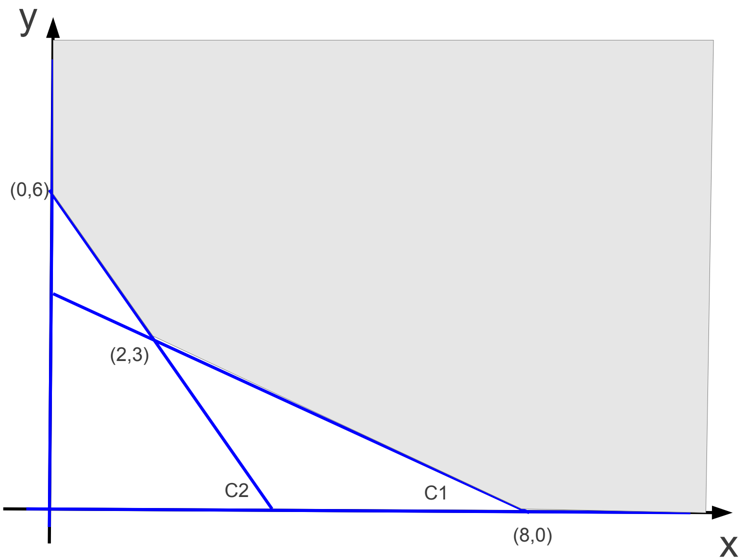
A minimization problem

min $x + 4y$

subject to:

$x + 2y \geq 8$
$3x + 2y \geq 12$
$x \geq 0$
$y \geq 0$

Graphical view



Step 1: Transformation

min $x + 4y$

subject to:

$x + 2y - s_1 = 8$
$3x + 2y - s_2 = 12$
$x \geq 0$
$y \geq 0$
$s_1, s_2 \geq 0$

Step 2: Matrix Form

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

If $x = y = 0$, then $s_1 = -8$, and $s_2 = -12$, which violates the nonnegativity restrictions on s_1, s_2.

Step 1: Transformation

Add artificial variables $\mathbf{a_1, a_2}$

min $x + 4y + 1000a_1 + 1000a_2$

subject to:

$$x + 2y - s_1 + a_1 = 8$$
$$3x + 2y - s_2 + a_2 = 12$$
$$x \geq 0$$
$$y \geq 0$$
$$s_1, s_2 \geq 0$$
$$a_1, a_2 \geq 0$$

Step 2: Matrix Form

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ 3 & 2 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

If $x = y = s_1 = s_2 = 0$, then $a_1 = 8$, and $a_2 = 12$, which gives us an easy starting point.

Simplex tableau:

p	x	y	s_1	s_2	a_1	a_2	Constant
1	-1	-4	0	0	-1000	-1000	0
0	1	2	-1	0	1	0	8
0	3	2	0	-1	0	1	12

Simplex tableau:

p	x	y	s_1	s_2	a_1	a_2	Constant	
1	-1	-4	0	0	-1000	-1000	0	$1000(R_2 + R_3) + R_1$
0	1	2	-1	0	1	0	8	
0	3	2	0	-1	0	1	12	

Step 3: Simplex Tableau

Simplex tableau:

p	x	y	s_1	s_2	a_1	a_2	Constant
1	3999	3996	-1000	-1000	0	0	20000
0	1	2	-1	0	1	0	8
0	3	2	0	-1	0	1	12

Step 4: Pivoting

For minimization, choose the column associated with largest positive element in R_1.

p	x	y	s_1	s_2	a_1	a_2	Constant
1	**3999**	3996	-1000	-1000	0	0	20000
0	**1**	2	-1	0	1	0	8
0	**3**	2	0	-1	0	1	12

Step 4: Pivoting

For minimization, choose the column associated with largest positive element in R_1.

p	x	y	s_1	s_2	a_1	a_2	Constant
1	3999	3996	-1000	-1000	0	0	20000
0	1	2	-1	0	1	0	8
0	**3**	2	0	-1	0	1	12

Step 5: Basis Update

p	x	y	s_1	s_2	a_1	a_2	Constant	
1	3999	3996	-1000	-1000	0	0	20000	$R_1 - 3999R_3/3$
0	1	2	-1	0	1	0	8	$R_2 - R_3/3$
0	**3**	2	0	-1	0	1	12	$R_3/3$

Step 5: Basis Update

p	x	y	s_1	s_2	a_1	a_2	Constant
1	0	1330	-1000	333	0	-1333	4004
0	0	4/3	-1	1/3	1	-1/3	4
0	1	2/3	0	-1/3	0	1/3	4

New basis:

$$\begin{array}{cc} a_1 & x \end{array}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $x = 4,\ y = 0,\ s_1 = 0,\ s_2 = 0,\ a_1 = 4,\ a_2 = 0$

Step 4: Pivoting

p	x	y	s_1	s_2	a_1	a_2	Constant
1	0	**1330**	-1000	333	0	-1333	4004
0	0	**4/3**	-1	1/3	1	-1/3	4
0	1	**2/3**	0	-1/3	0	1/3	4

Step 4: Pivoting

p	x	y	s_1	s_2	a_1	a_2	Constant
1	0	1330	-1000	333	0	-1333	4004
0	0	**4/3**	-1	1/3	1	-1/3	4
0	1	2/3	0	-1/3	0	1/3	4

Step 5: Basis Update

p	x	y	s_1	s_2	a_1	a_2	Constant	
1	0	1330	-1000	333	0	-1333	4004	$R_1 - 1330(3R_2/4$
0	0	**4/3**	-1	1/3	1	-1/3	4	$3R_2/4$
0	1	2/3	0	-1/3	0	1/3	4	$R_3 - 2/3(3R_2/4$

Step 5: Basis Update

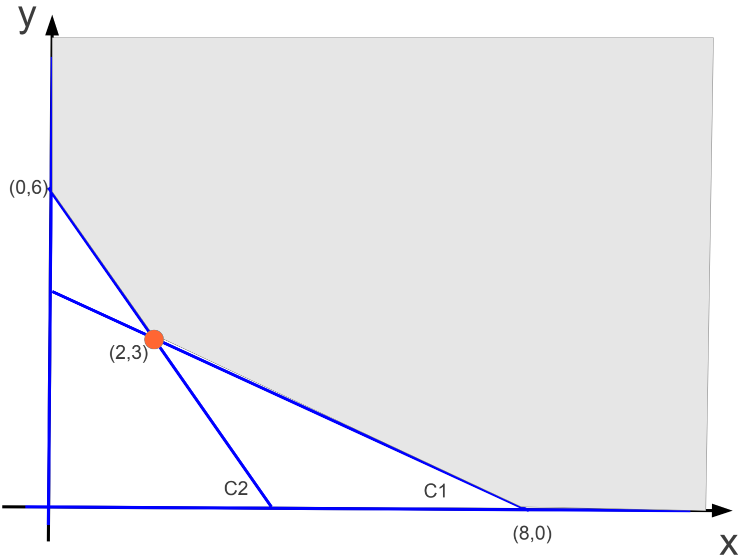
p	x	y	s_1	s_2	a_1	a_2	Constant
1	0	0	-5/2	1/2	-1995/2	-2001/2	14
0	0	1	-3/4	1/4	3/4	-1/4	3
0	1	0	-1/2	-1/2	-1/2	-1/6	2

New basis:

$$\begin{bmatrix} y & x \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $x = 2,\ y = 3,\ s_1 = 0,\ s_2 = 0,\ a_1 = 0,\ a_2 = 0$

Graphical view



Finish the example