University of Delaware

Department of Mathematical Sciences

MATH 243 – MIDTERM EXAM I – A – Spring 2014

Name:________________________ **Lecture Section:**_____**Discussion Section:**_____

Instructions:

• Check your examination booklet before you start.
There should be 7 items on 6 pages.

• No partial credit will be given if appropriate work is not shown.

• Answer questions in the space provided. If you need more space for an answer, continue your answer on the back of the page, or/and use the margins of the test pages. Do not take pages apart from the booklet.

• Carefully work out each problem and clearly indicate your final answer to any problem.

• You may **NOT** use calculators, dictionaries, notes, or any other kinds of aids.

• The duration of this exam is 75 minutes.

• **DISHONESTY WILL NOT BE TOLERATED:** Cheating receives a failing grade.

page 1	page 2	page 3	page 4	page 4	page 5	page 6	
q1	q2	q3	q4	q5	q6	q7	Total

(1) Let $\mathbf{a} = \langle 2, 2, 1\rangle$ and let $\mathbf{b} = \langle 1, 0, 1\rangle$.

(i) (5 points) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\vec{a} \circ \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$$

$$\cos\theta = \frac{\langle 2,2,1\rangle \circ \langle 1,0,1\rangle}{\sqrt{4+4+1}\,\sqrt{1+1}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

(ii) (5 points) Let $\mathbf{u} = \mathrm{Proj}_{\mathbf{b}}(\mathbf{a})$ be the vector projection of $\mathbf{a}$ onto $\mathbf{b}$. Find $\mathbf{u}$.

$$\vec{u} = \frac{\vec{a} \circ \vec{b}}{\|\vec{b}\|}\left(\frac{\vec{b}}{\|\vec{b}\|}\right) = \frac{3}{\sqrt{2}}\,\frac{\langle 1,0,1\rangle}{\sqrt{2}} = \left\langle \frac{3}{2}, 0, \frac{3}{2}\right\rangle$$

(iii) (5 points) Find a vector in $\mathbb{R}^3$ that is perpendicular to the vectors $\mathbf{c} = \langle 2, -1, 1\rangle$ and $\mathbf{d} = \langle 3, 1, -2\rangle$.

$$\vec{v} = \vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{vmatrix} = \langle 2-1, -(-4-3), 2+3\rangle = \langle 1, 7, 5\rangle$$

(2) Let L be the line with the following vector equation:

$$\mathbf{r}(t) = \langle 1 + 2t, 2 + 3t, 3 + 4t \rangle.$$

(i) (8 points) Determine whether L intersects the plane $x + y + z = 33$. If it does, find the intersection. If it does not, explain why not.

$(1+2t) + (2+3t) + (3+4t) = 33$

$6 + 9t = 33 \Rightarrow 9t = 27 \Rightarrow t = 3$

Intersection point:

$(1+2(3), 2+3(3), 3+4(3))$

$(7, 11, 15)$

(ii) (8 points) Determine whether L intersects the plane $3x + 2y - 3z = 2014$. If it does, find the intersection. If it does not, explain why not.

$3(1+2t) + 2(2+3t) - 3(3+4t) = 2014$

$3 + 6t + 4 + 6t - 9\cancel{t} - 12t = 2014$

$-2 = 2014$!! Contradiction

Our assumption of intersection cannot be true. In fact, the direction vector of the line is orthogonal to the normal vector of the plane, which means that the line is parallel to the plane.

(3) A particle moves along a curve with position at time t given by $\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle$.

(a) (12 points) Compute the velocity, speed, and acceleration of the particle at time t.

$\vec{v}(t) = \frac{d\vec{r}}{dt} = \langle 2, 2t, t^2 \rangle$

$\|\vec{v}(t)\| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2+2)^2} = |t^2+2| = t^2+2$

$\vec{a}(t) = \frac{d\vec{v}}{dt} = \langle 0, 2, 2t \rangle$

(b) (6 points) Find the length of the curve $\mathbf{r}(t)$ from $t = 1$ to $t = 2$.

$s = \int_1^2 \|\vec{r}\,'(t)\|\, dt = \int_1^2 (t^2+2)\, dt = \frac{t^3}{3} + 2t\Big|_1^2 = \left(\frac{8}{3}+4\right) - \left(\frac{1}{3}+2\right)$

$= \frac{7}{3} + 2 = \boxed{\frac{13}{3}}$

(4) (9 points) Find parametric equations for the tangent line to the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t^2 \rangle$ at the point $(0, 1, \frac{\pi^2}{4})$.

$\vec{r}'(t) = \langle -\sin t, \cos t, 2t \rangle$. At $(0, 1, \frac{\pi^2}{4})$, $t = \frac{\pi}{2}$

$\vec{r}'(\frac{\pi}{2}) = \langle -1, 0, \pi \rangle$

Line:

$\vec{w}(s) = \langle 0, 1, \frac{\pi^2}{4} \rangle + s\langle -1, 0, \pi \rangle = \langle -s, 1, \frac{\pi^2}{4} + s\pi \rangle$

(5) (9 points) A particle moves with velocity at time t given by $\mathbf{v}(t) = \langle e^{2t}, t, 2t \rangle$. If its position at time 0 is given by $\mathbf{r}(0) = \langle 3, -1, 2 \rangle$, find its position $\mathbf{r}(t)$ at time t.

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \frac{1}{2}e^{2t}, \frac{t^2}{2}, t^2 \rangle + \vec{C}$

$\vec{r}(0) = \langle \frac{1}{2}, 0, 0 \rangle + \vec{C} = \langle 3, -1, 2 \rangle \Rightarrow \vec{C} = \langle \frac{5}{2}, -1, 2 \rangle$

$\vec{r}(t) = \langle \frac{5}{2} + \frac{1}{2}e^{2t}, \frac{t^2}{2} - 1, t^2 + 2 \rangle$

(6) Let $\mathbf{r}(t) = \langle \cos(t) + t\sin(t), \sin(t) - t\cos(t), 0 \rangle$, defined for $t > 0$.

(i) (9 points) Find the unit tangent vector, $\hat{T}(t)$, for this curve.

$$\vec{r}\,'(t) = \langle -\sin t + \sin t + t\cos t,\ \cos t - \cos t + t\sin t, 0 \rangle$$

$$= \langle t\cos t,\ t\sin t, 0 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t| = t \quad \left(\text{because } t > 0\right)$$

$$\therefore\ \hat{T} = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle t\cos t, t\sin t, 0\rangle}{t} = \langle \cos t, \sin t, 0 \rangle$$

(ii) (9 points) Find the curvature, $\kappa(t)$, for this curve.

$$\kappa(t) = \frac{\|\hat{T}'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\sqrt{\cos^2 t + \sin^2 t}}{t} = \boxed{\frac{1}{t}}$$

(7) (15 points) Let $u(x, y) = \sin(x - y)$. Show that

$$u_{xxx} + u_{yyy} = 0.$$

$U_x = \cos(x-y)(1)$

$U_{xx} = -\sin(x-y)(1)$

$U_{xxx} = -\cos(x-y)(1) = -\cos(x-y)$.

$U_y = \cos(x-y)(-1) = -\cos(x-y)$

$U_{yy} = \sin(x-y)(-1) = -\sin(x-y)$

$U_{yyy} = -\cos(x-y)(-1) = \cos(x-y)$

$U_{xxx} + U_{yyy} = -\cos(x-y) + \cos(x-y) = 0$