University of Delaware

Department of Mathematical Sciences

MATH 243 – MIDTERM EXAM II – A – Spring 2014

Name: KEY **Lecture Section:**_____ **Discussion Section:**_____

Instructions:

- Check your examination booklet before you start.
There should be 8 items on 6 pages.

- No partial credit will be given if appropriate work is not shown.

- Answer questions in the space provided. If you need more space for an answer, continue your answer on the back of the page, or/and use the margins of the test pages. Do not take pages apart from the booklet.

- Carefully work out each problem and clearly indicate your final answer to any problem.

- You may **NOT** use calculators, dictionaries, notes, or any other kinds of aids.

- The duration of this exam is 75 minutes.

- **DISHONESTY WILL NOT BE TOLERATED:** Cheating receives a failing grade.

page 1 q1	page 1 q2	page 2 q3	page 3 q4	page 4 q5	page 4 q6	page 5 q7	page 6 q8	Total

(1) (10 points) Find an equation for the tangent plane to the surface $3xe^{z^2} + y^2 + z = 7$ at the point $(1, 2, 0)$.

$f(x,y,z) = 3xe^{z^2} + y^2 + z = 7$

$\nabla f(x,y,z) = \langle 3e^{z^2}, 2y, 6xze^{z^2} + 1 \rangle$

$\nabla f(1,2,0) = \langle 3, 4, 1 \rangle$

Plane

$3(x-1) + 4(y-2) + 1(z-0) = 0$

$3x + 4y - 11 = 0$

(2) (14 points) Given the function $f(x, y, z) = e^{xz} - y^2$ for the salinity of water at a point (x, y, z) in a body of water, answer the following questions.

(a) Find the rate of change of the salinity at the point $(0, 4, -1)$ when swimming in the direction toward the point $(2, 1, -7)$.

$\nabla f(x,y,z) = \langle ze^{xz}, -2y, xe^{xz} \rangle$

$\nabla f(0,4,-1) = \langle -1, -8, 0 \rangle$

$\vec{v} = \langle 2-0, 1-4, -7-(-1) \rangle = \langle 2, -3, -6 \rangle \quad \|\vec{v}\| = \sqrt{4+9+36} = 7$

$\hat{v} = \langle \frac{2}{7}, -\frac{3}{7}, -\frac{6}{7} \rangle \quad D_{\hat{v}} f(0,4,-1) = \langle -1, -8, 0 \rangle \cdot \langle \frac{2}{7}, -\frac{3}{7}, -\frac{6}{7} \rangle = \boxed{\frac{22}{7}}$

(b) Find the maximum rate of change of salinity at the point $(0, 4, -3)$.

$\nabla f(0,4,-3) = \langle -3, -8, 0 \rangle \Rightarrow$

$\|\nabla f(0,4,-3)\| = \sqrt{9+64} = \boxed{\sqrt{73}}$

(c) Find the direction of this maximum rate of change at the point $(0, 4, -3)$.

$\nabla f(0,4,-3) = \langle -3, -8, 0 \rangle$

(3) (14 points) Find the critical points of the function $f(x, y) = xy - x^2 - xy^2$ and classify them as local maximizers, local minimizers, or saddle points.

$$\nabla f(x,y) = \langle \underset{(1)}{y - 2x - y^2}, \underset{(2)}{x - 2xy} \rangle = \vec{0}$$

From (2) $x(1-2y) = 0 \Rightarrow$ $x = 0$ or $y = \frac{1}{2}$

$x = 0$ in (1): $y - y^2 = 0$
$y(1-y) = 0 \Rightarrow$ $y = 0$ or $y = 1$

Points: $(0,0)$, $(0,1)$

$y = \frac{1}{2}$ in (1): $\frac{1}{2} - 2x - \frac{1}{4} = 0 \Rightarrow \frac{1}{4} - 2x = 0 \Rightarrow x = \frac{1}{8}$

Point: $\left(\frac{1}{8}, \frac{1}{2}\right)$

$$H = \begin{pmatrix} -2 & 1-2y \\ 1-2y & -2x \end{pmatrix}$$

@ $(0,0)$

$H = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$

$\det(H) = -1 < 0$

$\therefore$ $(0,0)$ is a saddle point.

@ $(0,1)$

$H = \begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix}$

$\det(H) = -1 < 0$

$\therefore$ $(0,1)$ is a saddle point

@ $\left(\frac{1}{8}, \frac{1}{2}\right)$

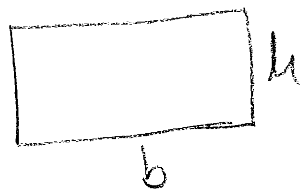
$H = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$

$\det(H) = \frac{1}{2} > 0$

& $-2 < 0$

$\therefore$ $\left(\frac{1}{8}, \frac{1}{2}\right)$ is a local maximizer.

(4) (14 points) Use Lagrange multipliers to find the height and width of the rectangle of maximum area whose perimeter is equal to 16. **You must use Lagrange multipliers to get credit on this problem.**



$f(b,h) = bh$

subject to $\underbrace{2b+2h}=16$

$\nabla f(b,h) = \lambda \nabla g(b,h)$

$\langle h, b \rangle = \lambda \langle 2, 2 \rangle \Rightarrow \left.\begin{array}{l} (1)\ h = 2\lambda \\ (2)\ b = 2\lambda \\ (3)\ 2b+2b=16 \end{array}\right\}$

(1) & (2) in (3):

$$2(2\lambda) + 2(2\lambda) = 16$$

$$8\lambda = 16 \Rightarrow \lambda = 2$$

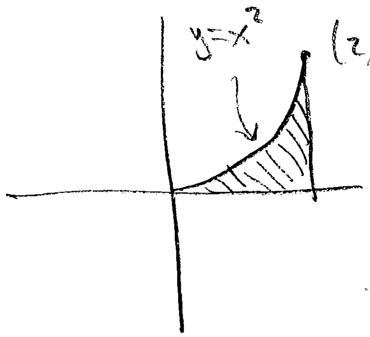
$\therefore$ $\boxed{\begin{array}{l} b = 2(2) = 4 \\ h = 2(2) = 4 \end{array}}$

The rectangle of maximum area with perimeter 16 is a 4x4 square.

(5) (10 points) Reverse the order of integration in the iterated integral $\int_0^4 \int_{\sqrt{y}}^2 3\sin(x^3)\,dx\,dy$.

That is, write this as an iterated integral in the order $dy\,dx$ with the appropriate bounds. You do **not** have to evaluate the resulting integral.

The region is $\left\{ \begin{array}{c} \sqrt{y} \le x \le 2 \\ 0 \le y \le 4 \end{array} \right\}$

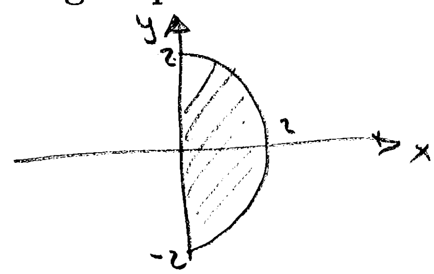


∴ Reversing bounds:

$$\int_0^2 \int_0^{x^2} 3\sin(x^3)\,dy\,dx$$

(6) (12 points) Evaluate $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x\,dy\,dx$ by **converting to polar coordinates.**

The region of integration is



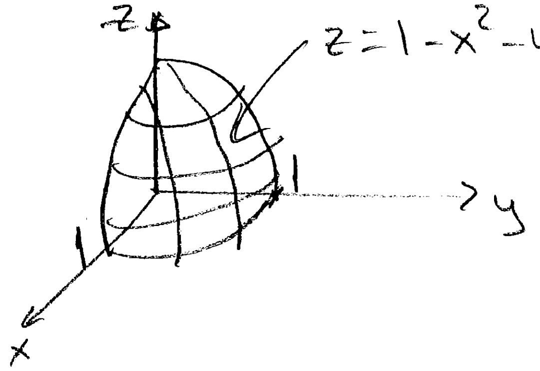
Using polar coordinates:

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x\,dy\,dx = \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^2 r\cos\theta\; r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 r^2\cos\theta\,dr\,d\theta =$$

$$\frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^3\Big|_0^2 \cos\theta\,d\theta = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,d\theta = \frac{8}{3}\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{\frac{16}{3}}$$

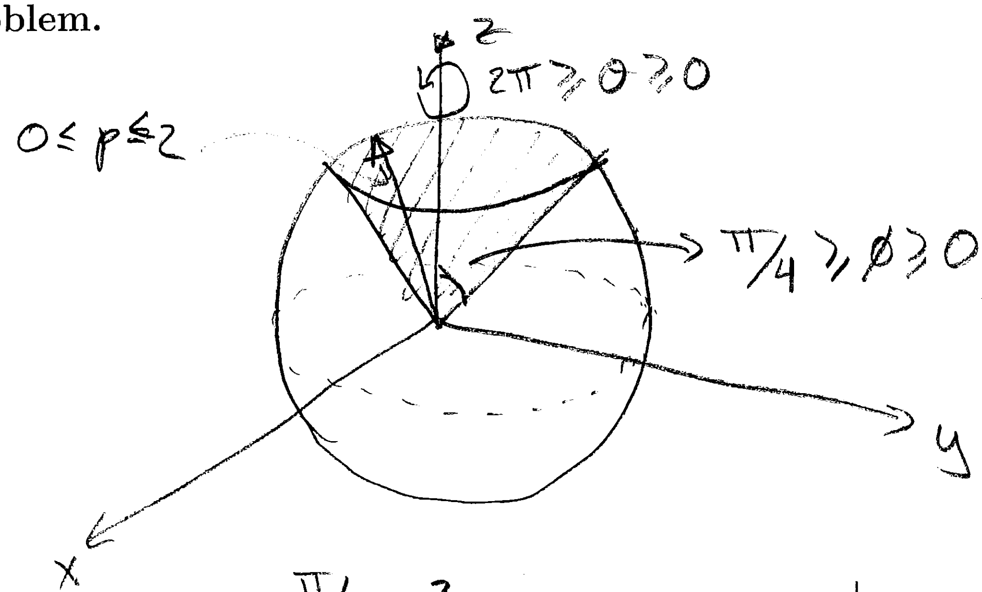
(7) (12 points) Let E be the solid in the first octant bounded by the paraboloid $z = 1 - x^2 - y^2$. (By the "first octant", we mean the region where x, y, and z are all nonnegative.) Using **cylindrical** coordinates, set up the triple integral (i.e., write an iterated integral with bounds) $\iiint_E (x+y)\,dV$. You do **not** have to evaluate this integral. **You must set up a triple integral in cylindrical coordinates to get any credit on this problem.**

$z = 1 - x^2 - y^2 = 1 - r^2$



$$\iiint_E (x+y)\,dV = \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r^2} (r\cos\theta + r\sin\theta)\, r\,dz\,dr\,d\theta$$

(8) (14 points) Let E be the solid in $\mathbb{R}^3$ bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the cone $z = \sqrt{x^2 + y^2}$. Use a **triple** integral in **spherical** coordinates to find the volume of E. **You must use spherical coordinates to get any credit on this problem.**



$$\iiint_E 1\, dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{\rho^3}{3}\Big|_0^2 \sin\phi \, d\phi \, d\theta$$

$$= \frac{8}{3} \int_0^{2\pi} -\cos\phi \Big|_0^{\pi/4} d\theta$$

$$= \frac{8}{3}\left(-\frac{\sqrt{2}}{2} + 1\right) \int_0^{2\pi} d\theta$$

$$= \frac{16}{3}\pi\left(1 - \frac{\sqrt{2}}{2}\right)$$