Name: Marco Montes de Oca
Section: 51

MATH 243 - Quiz 4
April 28, 2014

Please, SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Find and classify all the critical points of $f(x, y) = x^2 + y^2 + xy - x - y$.

$$\nabla f(x,y) = \langle 2x+y-1,\ 2y+x-1 \rangle = \vec{0}$$

$$\left.\begin{array}{l} 2x+y-1=0 \\ 2y+x-1=0 \end{array}\right\} \Rightarrow \begin{array}{l} y = 1-2x \\ 2(1-2x)+x-1=0 \\ 2-4x+x-1=0 \\ -3x+1=0 \Rightarrow x=\frac{1}{3}; \\ y = 1-2\left(\frac{1}{3}\right)=\frac{1}{3} \end{array}$$

Critical point: $\left(\frac{1}{3}, \frac{1}{3}\right)$

$$Hf(x,y) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow \begin{array}{l} 2>0 \\ \det(Hf(x,y)) = 2(2)-1=3>0 \end{array}$$

$\therefore \left(\frac{1}{3}, \frac{1}{3}\right)$ is a minimizer

2. (25 pts) Use Lagrange multipliers to find the extreme values of $f(x, y) = x^2 + y^2 + xy - x - y$ subject to $x + y = 1$.

$$\nabla f(x,y) = \langle 2x+y-1,\ 2y+x-1 \rangle = \lambda \langle 1, 1 \rangle$$

$$\left.\begin{array}{l} 2x+y-1=\lambda \\ 2y+x-1=\lambda \\ x+y=1 \end{array}\right\} \Rightarrow \begin{array}{l} 2x+y-1=2y+x-1 \\ \underline{x=y} \Rightarrow x+y=1 \\ \qquad 2x=1 \\ \qquad x=y=\frac{1}{2} \end{array}$$

So an extreme value of f is

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}+\frac{1}{4}+\frac{1}{4}-\frac{1}{2}-\frac{1}{2} = \frac{3}{4}-\frac{2}{2} = \boxed{-\frac{1}{4}}$$

3. (25 pts) Calculate the average value of $f(x,y) = x^2 + y^2$ over the unit disk $x^2 + y^2 \leq 1$. (The average value of a function $f(x,y)$ over a region D is defined by $\dfrac{\iint_D f(x,y)\,dA}{\text{Area of D}}$.)

Since D is a disk we use polar coordinates:

$f(r,\theta) = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

$$\iint_D f(x,y)\,dA = \int_0^{2\pi}\int_0^1 r^2(r)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^1 d\theta = \frac{1}{4}\int_0^{2\pi} d\theta = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$\therefore \text{Avg. value} = \frac{\frac{\pi}{2}}{\pi} = \frac{1}{2}$$

area of D $\to$ π

4. (25 pts) Set up a triple integral in spherical coordinates to find the volume that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cone $z = \sqrt{x^2 + y^2}$ on the first octant.

$\rho^2 = 2 \Rightarrow \rho = \sqrt{2}$

$\tan\phi = 1 \Rightarrow \phi = \frac{\pi}{4}$

$$\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} \rho^2\sin\phi \,d\rho\,d\phi\,d\theta$$