Name: Marco A. Montes de Oca
Section: 51

MATH 243 - Quiz 45
May 14, 2014

Please, SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Evaluate the line integral $\int_C 2x\,ds$, where C is the curve along the graph of $y = x^2$ from (0,0) to (2,4).

C: $\vec{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 2$.

$\vec{r}\,'(t) = \langle 1, 2t \rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{1+4t^2}$

$$\int_C 2x\,ds = \int_0^2 2(t)\sqrt{1+4t^2}\,dt = \frac{1}{4}\int_1^{17} \sqrt{u}\,du = \frac{1}{4}\cdot\frac{2}{3}(u)^{3/2}\Big|_1^{17} = \boxed{\frac{1}{6}\left(17^{3/2}-1\right)}$$

$u = 1+4t^2$
$du = 8t\,dt$

2. (25 pts) Find the work done by the force field $\vec{F}(x,y) = \langle 2xy, x^2+2y \rangle$ on a particle whose trajectory is described by $C : \vec{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 2$.

$P = 2xy$, $Q = x^2 + 2y$

$Q_x = 2x$, $P_y = 2x$ $\therefore$ $\vec{F}$ is conservative and is defined everywhere in $\mathbb{R}^2$. Since we can use the Fundamental Theorem of Calculus for Line Integrals:

$f_c = \int P\,dx = \int 2xy\,dx = x^2y + C(y) \Rightarrow \frac{\partial f_c}{\partial y} = x^2 + C'(y) \stackrel{?}{=} x^2 + 2y$

$C'(y) = 2y \Rightarrow C(y) = y^2 + K$ $\therefore$ f such that $\nabla f = \vec{F}$ is

$f(x,y) = \underline{x^2y + y^2 + K}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(0)) = f(2,4) - f(0,0) = 32 + K - (K)$$

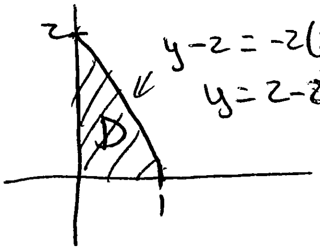
$$= 32$$

3. (25 pts) Calculate the work done by the force field $\vec{F}(x,y) = \langle xy, x^2 \rangle$ on a particle whose trajectory starts at (0,0) continues to (1,0), then moves to (0,2) and returns to (0,0).

$P = xy$, $Q = x^2 \Rightarrow Q_x = 2x$, $P_y = x \Rightarrow \vec{F}$ is not conservative.

Since C is closed and positively defined, we can use Green's Theorem with $Q_x - P_y = 2x - x = x$, so

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D x \, dA = \int_0^1 \int_0^{2-2x} x \, dy \, dx = \int_0^1 xy \Big|_0^{2-2x} dx = \int_0^1 x(2-2x)\,dx =$$



$y - 2 = -2(x - 0)$

$y = 2 - 2x$

$$\int_0^1 (2x - 2x^2)\,dx = x^2 - \frac{2x^3}{3} \Big|_0^1 = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$$

4. (25 pts) Calculate the surface area of the part of the surface $f(x,y) = xy$ over the disk $x^2 + y^2 <= 1$.

Surface: $\vec{r}(x,y) = \langle x, y, xy \rangle \Rightarrow \vec{r}_x(x,y) = \langle 1, 0, y \rangle$, $\vec{r}_y(x,y) = \langle 0, 1, x \rangle \Rightarrow \vec{r}_x \times \vec{r}_y = \langle -y, -x, 1 \rangle$

$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{1 + x^2 + y^2}$

$SA = \iint_M \sqrt{1 + x^2 + y^2}\, dA$, where M is unit disk.

Using polar coordinates:

$$SA = \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}\, r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_1^2 \sqrt{u}\, du \, d\theta = \frac{1}{2} \int_0^{2\pi} \frac{2}{3} u^{3/2} \Big|_1^2 d\theta$$

$u = 1 + r^2$

$du = 2r\,dr$

$$= \frac{1}{3} \int_0^{2\pi} \left((2)^{3/2} - 1 \right) d\theta = \boxed{\left(\frac{\sqrt{8} - 1}{3} \right)(2\pi)}$$