University of Delaware
Department of Mathematical Sciences

MATH-529 – Fundamentals of Optimization
Instructor: Dr. Marco A. Montes de Oca
Spring 2014

Exam I

Name: Marco Montes de Oca **Section:** 10

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Question	1	2	3	4	5	Total
Points						

Instructions

- The exam consists of **five** problems for a total of 100 points.
- Read very carefully each problem before working on it.
- Partial credit will not be given if appropriate work is not shown.
- If you get stuck on a problem, skip it and come back to it if you have extra time at the end.
- Answer questions in the space provided. If you need more space for an answer, continue your answer on the back of the page, or/and use the margins of the test pages.
- Carefully work out each problem and clearly indicate your final answer to any problem.
- **You may use calculators** but no other aids such as dictionaries, notes, books, etc.
- **DISHONESTY WILL NOT BE TOLERATED.**

Problems

1. [20 points] Let $f(x,y) = x^3 - axy + by^3$, where $a \neq 0$ and $b \neq 0$.

a) (5 points) Find the critical points of f.

$$\nabla f(x,y) = \begin{pmatrix} 3x^2 - ay \\ 3by^2 - ax \end{pmatrix} = \vec{0} \Rightarrow$$

$$3x^2 = ay \Rightarrow y = \frac{3x^2}{a}$$

In (2):

$$3b\left(\frac{3x^2}{a}\right)^2 - ax = 0$$

$$3b\frac{9x^4}{a^2} - ax = 0$$

$$x\left(\frac{27bx^3}{a^2} - a\right) = 0$$

$\therefore$ $x = 0$ or

$$\frac{27bx^3}{a^2} - a = 0 \Rightarrow$$

$$27bx^3 = a^3 \Rightarrow x^3 = \frac{a^3}{27b}$$

$$x = \frac{a}{3b^{1/3}}$$

Points:

$(x=0,\ y=0)$

$\left(x = \frac{a}{3b^{1/3}},\ y = \frac{a}{3b^{2/3}}\right)$

b) (5 points) Show that one of the critical points you found in a) is a saddle point independently of the values of a and b.

$$Hf(x,y) = \begin{pmatrix} 6x & -a \\ -a & 6by \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} 0 & -a \\ -a & 0 \end{pmatrix} \Rightarrow \det(Hf(0,0)) = -a^2 < 0$$

$\therefore$ $(0,0)$ is a saddle point.

c) (5 points) For what values of a and b is the other critical point you found in a) a strict local minimizer of f? For what values of a and b is this point a strict local maximizer?

$$Hf\left(\frac{a}{3\,b^{1/3}}, \frac{a}{3\,b^{2/3}}\right) = \begin{pmatrix} \frac{2a}{b^{1/3}} & -a \\ -a & 2a\,b^{1/3} \end{pmatrix}$$

$$\Delta_1 = \frac{2a}{b^{1/3}}$$

$$\Delta_2 = 4a^2 - a^2 = 3a^2$$

Since $\Delta_2 > 0$, the nature of the critical point depends on the sign of Δ_1. For a local minimizer, we need $\Delta_1 > 0$ which occurs when $a>0$ & $b>0$ or $a<0$ & $b<0$. For a local maximizer, we need $\Delta_1 < 0$ which occurs when $a>0$ & $b<0$ or $a<0$ & $b>0$

d) (5 points) Show that f has no global minimizers or maximizers.

If $y=0$, then $f \to \pm\infty$ as $x \to \pm\infty$, or if $x=0$, then $f \to \pm\infty$ as $y \to \pm\infty$. This shows that f is unbounded below and above and therefore f has no global minimizers or maximizers.

2. [20 points] Determine the values of a, b, and c that guarantee that $f(x, y) = 4x^2 + axy + y^2 + bx + 3y + c$ is convex.

$$\nabla f(x,y) = \begin{pmatrix} 8x + ay + b \\ ax + 2y + 3 \end{pmatrix}$$

$$Hf(x,y) = \begin{pmatrix} 8 & a \\ a & 2 \end{pmatrix}$$

If f is convex, its Hessian should be positive semidefinite. Therefore; we need

$$\Delta_1 = 8 > 0$$

$$\Delta_2 = 16 - a^2 \geq 0$$

$$16 \geq a^2 \Rightarrow |a| \leq 4$$

$$\underline{-4 \leq a \leq 4}$$

$b \in \mathbb{R}$ and

$c \in \mathbb{R}$

3. [20 points] Suppose you use Newton's method with Hessian modification to find a minimizer of Rosenbrock's function, which is defined by $f(x, y) = (1-x)^2 + 100(y - x^2)^2$. Find a starting point for the method so that the procedure for the eigenvalue modification of the Hessian is called during the method's first iteration.

$$\nabla f(x,y) = \begin{pmatrix} -2(1-x) - 400x(y-x^2) \\ 200(y-x^2) \end{pmatrix}$$

$$Hf(x,y) = \begin{pmatrix} 2 - 400y + 1200x^2 & -400x \\ -400x & 200 \end{pmatrix}$$

The eigenvalue decomposition is called if Hf is not positive definite. This will happen, for instance at (0,1) where the Hessian is

$$Hf(0,1) = \begin{pmatrix} 2-400 & 0 \\ 0 & 200 \end{pmatrix} = (-398)(200) < 0$$

4. [20 points] Follow the instructions below.

a) (10 points) Find the extreme points of $f(x, y) = x + 2y$ subject to $xy = 1$. By comparing the corresponding objective function values of the points you find, identify which point is a maximizer and which point is a minimizer.

$$L(x,y) = x + 2y + \lambda(1 - xy)$$

$$\nabla L(x,y) = \begin{pmatrix} 1 - \lambda y \\ 2 - \lambda x \\ 1 - xy \end{pmatrix} = 0 \Rightarrow \begin{matrix} 1 = \lambda y & (1) \\ 2 = \lambda x & (2) \\ 1 = xy & (3) \end{matrix}$$

(1) & (2) in (3):

$$\left(\frac{2}{\lambda}\right)\left(\frac{1}{\lambda}\right) = 1 \Rightarrow 2 = \lambda^2 \Rightarrow \lambda = \pm\sqrt{2}$$

$$\therefore \left.\begin{matrix} x = \pm \frac{2}{\sqrt{2}} \\ y = \pm \frac{1}{\sqrt{2}} \end{matrix}\right\} \text{points: } \left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{2}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$f\left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} + 2\left(\frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}} \quad \therefore \left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ is a maximizer.}$$

$$f\left(-\frac{2}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} + 2\left(-\frac{1}{\sqrt{2}}\right) = -\frac{4}{\sqrt{2}} \quad \therefore \left(-\frac{2}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \text{ is a minimizer.}$$

b) (10 points) Consider the point $(10^{-10}, 10^{10})$, which satisfies the constraint $xy = 1$. This point has a higher objective function value than the maximizer of the previous point. Explain why the Lagrange method does not detect this (and other similar) solution(s).

The method fails because the assumptions typically used for constraints (continuous and differentiable) does not hold at (0,0).

5. [20 points] The code excerpt below shows the main loop of a basic implementation of the steepest descent method. Explain what the highlighted section does and why it is important.

```
for i=1:1:maxiter
  if i==1
    [current,g,H] = hard(x(1),x(2));
    best=current;
    solution = x;
  else

    d = -g/norm(g);

    alpha = a0;
    while true
        xc = x + alpha.*d;
        [candidate,cg,cH] = hard(xc(1),xc(2));
        [current,g,H] = hard(x(1),x(2));
        if candidate >  current + c*alpha*cg*d';
            alpha = rho*alpha;
        else
            break;
        end
    end

    x = x + alpha.*d;

    [current,g,H] = hard(x(1),x(2));

    if current < best
      best=current;
      solution = x;
    end
  end

  points(i,1)=x(1);
  points(i,2)=x(2);
  disp(['Iteration =' num2str(i) ' Obj. function value = '
      num2str(current) ' Current point = ' num2str(x) ]);
end
```

The highlighted section implements the backtracking method to calculate a step length that satisfies the Wolfe conditions of sufficient descent and curvature.