University of Delaware
Department of Mathematical Sciences

MATH-529 – Fundamentals of Optimization
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Spring 2014

Exam II

Name: KEY **Section:** 10

May 7, 2014

Question	1	2	3	4	5	Total
Points						

Instructions

- The exam consists of **five** problems for a total of 100 points.
- Read very carefully each problem before working on it.
- Partial credit will not be given if appropriate work is not shown.
- If you get stuck on a problem, skip it and come back to it if you have extra time at the end.
- Answer questions in the space provided. If you need more space for an answer, continue your answer on the back of the page, or/and use the margins of the test pages.
- Carefully work out each problem and clearly indicate your final answer to any problem.
- **You may use calculators** but no other aids such as dictionaries, notes, books, etc.
- **DISHONESTY WILL NOT BE TOLERATED.**

Problems

1. [20 points] Follow the instructions below:

a) (10 points) Using the Lagrange multipliers method, find the extreme point(s) of the constrained problem

$$\text{Optimize } x^4 - x^2 + y^4 - y^2$$
$$\text{subject to } x - y = 0$$

$$L(x,y,\lambda) = x^4 - x^2 + y^4 - y^2 + \lambda(y-x)$$

$$\nabla L(x,y,\lambda) = \begin{pmatrix} 4x^3 - 2x - \lambda \\ 4y^3 - 2y + \lambda \\ y - x \end{pmatrix} = \vec{0}$$

From (3) $x=y$;

In (1)+(2):

$$8x^3 - 4x = 0$$
$$x(8x^2-4) = 0$$
$$x = 0 \ ; \ y = 0$$
or
$$x = \pm\frac{1}{\sqrt{2}} \ ; \ y = \pm\frac{1}{\sqrt{2}}$$

Points: $(0,0)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

b) (10 points) Using a bordered Hessian, classify the point(s) found in the previous step as local maximizer(s) or minimizer(s).

$$\bar{H} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 12x^2-2 & 0 \\ -1 & 0 & 12y^2-2 \end{pmatrix}$$

@$(0,0)$

$$\bar{H} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix}$$

Since $\det(\bar{H}) = -1(-2) - 1(-2) = +4 > 0$

$(0,0)$ is a local MAXIMIZER

@$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\bar{H} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

Since $\det(\bar{H}) = -1(4) - 1(4) = -8 < 0$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a local MINIMIZER

@$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$$\bar{H} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

$\det(\bar{H}) = -8 < 0$

$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is a local MINIMIZER.

2. [20 points] Follow the instructions below:

a) (10 points) Find all the points that satisfy the KKT conditions for the following program:

$$\text{Minimize } y - x$$
$$\text{subject to } y - x^2 \geq 0$$
$$2 - x - y \geq 0$$

$$L(x,y,\lambda_1,\lambda_2) = y - x + \lambda_1(x^2 - y) + \lambda_2(x + y - 2)$$

$$\nabla_{x,y} L(x,y,\lambda_1,\lambda_2) = \begin{pmatrix} -1 + 2\lambda_1 x + \lambda_2 \\ 1 - \lambda_1 + \lambda_2 \end{pmatrix} = \vec{0}$$

$y - x^2 \geq 0$, $2 - x - y \geq 0$

$\lambda_1(y - x^2) = 0$, $\lambda_2(2 - x - y) = 0$

$\lambda_1, \lambda_2 \geq 0$

KKT

$\lambda_1 = \lambda_2 = 0$	$\lambda_1 = 0,\ \lambda_2 > 0$	$\lambda_1 > 0\ \lambda_2 = 0$	$\lambda_1 > 0\ \lambda_2 > 0$
Violates (1) & (2) X	Violates (2) X	From (2) $\lambda_1 = 1$ In (1): $x = \frac{1}{2}$ From (5): $y = x^2 = \frac{1}{4}$ $\left(\frac{1}{2}, \frac{1}{4}\right)$ $\lambda_1 = 0, \lambda_2 = 0$	From (5) & (6): $y = x^2$; $x + y = 2$ $x^2 + x - 2 = 0 \Rightarrow (x-1)(x+2) = 0$ $(x=1, y=1)$ a) $(x=-2, y=4)$ b)

b) (10 points) Determine whether LICQ or MFCQ are satisfied at the solution point(s).

Active constraint at $\left(\frac{1}{2}, \frac{1}{4}\right)$: c_1

$$\nabla c_1 = \begin{pmatrix} -2x \\ 1 \end{pmatrix} \text{ @ } \left(\tfrac{1}{2}, \tfrac{1}{4}\right): \nabla c_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Since $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is linearly independent, then LICQ is satisfied.

If, for example, $w = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, then $w^T \nabla c = 3 > 0$

$\therefore$ MFCQ is satisfied too.

a) In (1) & (2):

$-1 + 2\lambda_1 + \lambda_2 = 0$

$1 - \lambda_1 + \lambda_2 = 0$

$\lambda_1 = \lambda_2 + 1$

$-1 + 2(\lambda_2 + 1) + \lambda_2 = 0$

$-1 + 2\lambda_2 + 2 + \lambda_2 = 0$

$3\lambda_2 + 1 = 0$ $\lambda_2 < 0$

Violates (7)

b) In (1) & (2):

$-1 - 4\lambda_1 + \lambda_2 = 0$

Since $\lambda_1 = \lambda_2 + 1$:

$-1 - 4(\lambda_2 + 1) + \lambda_2 = 0$

$-3\lambda_2 - 5 = 0$

$\lambda_2 < 0$

Violates (7)

3. [20 points] Follow the following instructions:

a) (10 points) Form a quadratic penalized function for the program

$$\text{Minimize } xy$$

$$\text{subject to } 1 + y - x = 0.$$

$$Q(x,y,\mu) = xy + \frac{\mu}{2}(1+y-x)^2$$

b) (10 points) Show that the penalized function that you defined above is unbounded below if $\mu < \frac{1}{2}$. (In other words, show that when $\mu < \frac{1}{2}$, the penalized function does not have a global minimum.)

To show that Q is unbounded below when $\mu < \frac{1}{2}$, we need to show that the Hessian of Q when $\mu < \frac{1}{2}$ is not positive (semi) definite.

$$\nabla Q(x,y,\mu) = \begin{pmatrix} y + \mu(1+y-x)(-1) \\ x + \mu(1+y-x) \end{pmatrix}$$

$$HQ(x,y,\mu) = \begin{pmatrix} \mu & 1-\mu \\ 1-\mu & \mu \end{pmatrix}$$

We need $\Delta_1 > 0$ & $\Delta_2 \geq 0$ for HQ to be positive definite

thus: $\mu > 0$ & $\mu^2 - (1-\mu)^2 \geq 0$

$$\mu^2 - (1 - 2\mu + \mu^2) \geq 0$$

$$\cancel{\mu^2} - 1 + 2\mu - \cancel{\mu^2} \geq 0$$

$$2\mu \geq 1$$

$$\mu \geq \frac{1}{2} \therefore$$

$\boxed{\text{If } \mu < \frac{1}{2} \text{ a critical point of } Q \text{ would be a saddle point and therefore } Q \text{ would be unbounded below.}}$

4. [20 points] Follow the following instructions:

a) (10 points) Form the augmented Lagrangian associated with the program

$$\text{Minimize } x^2$$

$$\text{subject to } c(x) = x - 1 = 0\,.$$

$$L(x,\lambda,\mu) = x^2 + \lambda(1-x) + \frac{\mu}{2}(x-1)^2$$

b) Simulate two iterations of the augmented Lagrangian method starting with $\lambda_0 = 0$ and $\mu_0 = 1$ (At each iteration solve the resulting unconstrained optimization problem exactly, and update λ and μ using $\lambda_{k+1} = \lambda_k - \mu_k c(x_k)$, and $\mu_{k+1} = \alpha\mu_k$). Use $\alpha = 2$.

Iteration 1:

$$L(x,0,1) = x^2 + \frac{1}{2}(x-1)^2$$

$$\nabla L(x,0,1) = 2x + (x-1) = 3x - 1 = 0 \Rightarrow x_1 = \frac{1}{3}$$

$$\lambda_1 = 0 - 1\left(\frac{1}{3} - 1\right) = -1\left(-\frac{2}{3}\right) = \frac{2}{3}$$

$$\mu_1 = 2(1) = 2$$

Iteration 2:

$$L\left(x,\frac{2}{3},2\right) = x^2 + \frac{2}{3}(1-x) + (x-1)^2$$

$$\nabla L\left(x,\frac{2}{3},2\right) = 2x - \frac{2}{3} + 2(x-1) = 4x - \frac{2}{3} - 2 = 4x - \frac{8}{3} = 0$$

$$\Rightarrow x_2 = \frac{\frac{8}{3}}{4} = \frac{2}{3}$$

$$\lambda_2 = \frac{2}{3} - 2\left(\frac{2}{3} - 1\right) = \frac{2}{3} - \frac{4}{3} + 1 = -\frac{2}{3} + \frac{3}{3} = \frac{1}{3}$$

$$\mu_2 = 2(2) = 4$$

5. [20 points] Determine the dual function of the program:

$$\text{Minimize } x+y$$
$$\text{subject to } 2x - x^2 - y^2 \geq 0.$$

[Simplify as much as you can.]

$$L(x,y,\lambda) = x+y+\lambda(x^2+y^2-2x)$$

$$\nabla_{x,y} L(x,y,\lambda) = \begin{pmatrix} 1+2\lambda x - 2\lambda \\ 1+2\lambda y \end{pmatrix}$$

$$HL(x,y,\lambda) = \begin{pmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{pmatrix}$$

Since $\lambda \geq 0$ by the KKT conditions, a critical point of L is a minimizer and therefore the minimum of L is the infimum of L wrt x, y.

$\nabla_{x,y} L(x,y,\lambda) = \vec{0}$ implies

$$2\lambda x = 2\lambda - 1 \Rightarrow x = \frac{2\lambda - 1}{2\lambda} \quad \text{assuming } \lambda > 0$$
$$2\lambda y = -1 \Rightarrow y = -\frac{1}{2\lambda}$$

Back in L:

$$L_d(\lambda) = \frac{2\lambda-1}{2\lambda} - \frac{1}{2\lambda} + \lambda\left(\left(\frac{2\lambda-1}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 - 2\left(\frac{2\lambda-1}{2\lambda}\right)\right)$$

$$= 1 - \frac{1}{2\lambda} - \frac{1}{2\lambda} + \lambda\left(\left(1-\frac{1}{2\lambda}\right)^2 + \frac{1}{4\lambda^2} - \frac{2\lambda-1}{\lambda}\right)$$

$$= 1 - \frac{1}{\lambda} + \lambda\left(1 - \cancel{\frac{1}{\lambda}} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} - 2 + \cancel{\frac{1}{\lambda}}\right)$$

$$= 1 - \frac{1}{\lambda} + \lambda\left(\frac{1}{2\lambda^2} - 1\right) = 1 - \frac{1}{\lambda} + \frac{1}{2\lambda} - \lambda = \frac{2\lambda - 2 + 1 - 2\lambda^2}{2\lambda}$$

$$= \boxed{\frac{-2\lambda^2 + 2\lambda - 1}{2\lambda}}$$

If $\lambda = 0$, the problem does not have a solution because $x+y$ is unbounded.