HW1 M529 - Spring 14

Solutions

1. $f(x) = x^4 - 2x^3 + x + 1$

$f'(x) = 4x^3 - 6x^2 + 1$, if $f'(x^*) = 0$ then

$$4(x^*)^3 - 6(x^*)^2 + 1 = 0 \Rightarrow \left.\begin{array}{l} x^* = \frac{1}{2} \\ x^* = \frac{1}{2}(1-\sqrt{3}) \\ x^* = \frac{1}{2}(1+\sqrt{3}) \end{array}\right\} \text{Critical Points}$$

Local minimizers, maximizers.

$f''(x) = 12x^2 - 12x$

For $x^* = \frac{1}{2}$:

$f''\left(\frac{1}{2}\right) = -3 < 0 \therefore$ x^* is a local maximizer

For $x^* = \frac{1}{2}(1-\sqrt{3})$:

$f''\left(\frac{1}{2}(1-\sqrt{3})\right) = 6 > 0 \therefore$ x^* is a local minimizer

For $x^* = \frac{1}{2}(1+\sqrt{3})$:

$f''\left(\frac{1}{2}(1+\sqrt{3})\right) = 6 > 0 \therefore$ x^* is a local minimizer

Global minimizers, maxmizers.

It is clear from the previous analysis that $f''(x)$ changes sign. Therefore the 2nd order derivative does not provide information regarding the global nature of x^*.

However, we note that

$$\lim_{x \to \pm\infty} f(x) = +\infty$$

So we conclude that

$$x^* = \frac{1}{2}(1 \pm \sqrt{3}) \quad \text{are global minimizers}$$

while $x^* = \frac{1}{2}$ is just a local maximizer.

$\therefore \quad f(x) = \sin\left(\frac{1}{x}\right)$

$f'(x) = \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$, we must assume $x \neq 0$.

Now, to find critical points:

$$f'(x^*) = -\frac{1}{(x^*)^2}\cos\left(\frac{1}{x^*}\right) = 0 \Rightarrow \cos\left(\frac{1}{x^*}\right) = 0$$

$$\therefore \quad \frac{1}{x^*} = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots\right\}$$

or

$$\frac{1}{x^*} = (1+2n)\frac{\pi}{2}, \quad n \in \mathbb{Z} \Rightarrow x^* = \frac{2}{\pi(1+2n)}, \quad n \in \mathbb{Z}.$$

Since $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \ \forall x \neq 0$, it is evident that all

$x^* = \frac{2}{\pi(1+2n)}$, for all even n, are local and global maximizers

and

$x^* = \frac{2}{\pi(1+2n)}$, for all odd n, are local and global minimizers.

3. $f(x) = 3x^5 - 10x^3 + 15x + 30$

$f'(x) = 15x^4 - 30x^2 + 15$

Critical points:

$$f'(x^*) = 15(x^*)^4 - 30(x^*)^2 + 15 = 0$$

$$(x^*)^4 - 2(x^*)^2 + 1 = 0$$

$$[(x^*)^2 - 1]^2 = 0 \Rightarrow x^* = \pm 1$$

Local optimizers:

$$f''(x) = 60x^3 - 60x$$

$$\left.\begin{array}{l} f''(1) = 60 - 60 = 0 \\ f''(-1) = -60 + 60 = 0 \end{array}\right\}$$

x^* can be either a local minimizer, maximizer or an inflection point.

In this particular case, $x^* = \pm 1$, are inflection points but this is only seen by inspection of a neighborhood of x^*.

Global optimizers:

Since $f(x)$ is not bounded below or above, that is $\lim_{x \to \pm\infty} f(x) = \pm\infty$, $f(x)$ does not have global minimizers or maximizers.

4. $f(\vec{x}) = x_1^3 - x_1^2 x_2 - x_1 x_2 + x_2^2$

$$\nabla f(\vec{x}) = \begin{pmatrix} 3x_1^2 - 2x_1 x_2 - x_2 \\ -x_1^2 - x_1 + 2x_2 \end{pmatrix}$$

Critical points:

$$\nabla f(\vec{x}^*) = 0 = \begin{pmatrix} 3(x_1^*)^2 - 2(x_1^*)(x_2^*) - x_2^* \\ -(x_1^*)^2 - x_1^* + 2x_2^* \end{pmatrix}$$

$$\vec{x}^* = (0,0),\ \left(\tfrac{1}{2}, \tfrac{3}{8}\right),\ (1,1)$$

Local optimizers:

$$Hf(\vec{x}) = \begin{pmatrix} 6x_1 - 2x_2 & -2x_1 - 1 \\ -2x_1 - 1 & 2 \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} \Rightarrow \det(Hf(0,0)) = -1 < 0$$

so $(0,0)$ is a saddle point.

$$Hf\left(\tfrac{1}{2}, \tfrac{3}{8}\right) = \begin{pmatrix} 3 - \frac{3}{4} & -2 \\ -2 & 2 \end{pmatrix} \Rightarrow \det\left(Hf\left(\tfrac{1}{2}, \tfrac{3}{8}\right)\right) = \frac{9}{2} - 4 > 0$$

So $\left(\tfrac{1}{2}, \tfrac{3}{8}\right)$ is a local minimizer

$$Hf(1,1) = \begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix} \Rightarrow \det(Hf(1,1)) = -1 < 0$$

So $(1,1)$ is a saddle point

Global optimizers:

We just saw that $Hf(\vec{x})$ is indefinite in $\mathbb{R}^2$.
This means that $\left(\frac{1}{2}, \frac{3}{8}\right)$ might be a global minimizer but 2nd order information is inconclusive.

Inspecting $f(\vec{x})$ more closely, we notice that if $x_2 = 0$, $f(x_1, 0)$ is unbounded, which means that $f(\vec{x})$ does not have global minimizers or maximizers.

5. $f(\vec{x}) = x_1^2 + x_2^2 - 2x_1x_2$

$$\nabla f(\vec{x}) = \begin{pmatrix} 2x_1 - 2x_2 \\ 2x_2 - 2x_1 \end{pmatrix}$$

$$\nabla f(\vec{x}^*) = \begin{pmatrix} 2x_1^* - 2x_2^* \\ 2x_2^* - 2x_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1^* = x_2^*$$

There are infinitely many critical points.

Local optimizers:

$$Hf(\vec{x}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \Rightarrow \det(Hf(\vec{x})) = 0$$

$\therefore$ $Hf(\vec{x})$ is positive semidefinite $\forall \vec{x} \in \mathbb{R}^2$ and therefore all the critical points are local minimizers as well as global minimizers.

Global minimizers: See above.

6. $f(\vec{x}) = e^{x_1 - x_2} + e^{x_2 - x_1}$

$$\nabla f(\vec{x}) = \begin{pmatrix} e^{x_1 - x_2} - e^{x_2 - x_1} \\ e^{x_2 - x_1} - e^{x_1 - x_2} \end{pmatrix}$$

$$\nabla f(\vec{x}^*) = \begin{pmatrix} e^{x_1^* - x_2^*} - e^{x_2^* - x_1^*} \\ e^{x_2^* - x_1^*} - e^{x_1^* - x_2^*} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1^* = x_2^*$$

There are infinitely many critical points.

Local optimizers:

$$Hf(\vec{x}) = \begin{pmatrix} e^{x_1 - x_2} + e^{x_2 - x_1} & -e^{x_1 - x_2} - e^{x_2 - x_1} \\ -e^{x_2 - x_1} - e^{x_1 - x_2} & e^{x_2 - x_1} + e^{x_1 - x_2} \end{pmatrix}$$

$$Hf(\vec{x}^*) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \Rightarrow \det(Hf(\vec{x}^*)) = 0$$

$\therefore$ All critical points are local minimizers because $Hf(\vec{x}^*)$ is positive semidefinite.

Global optimizers:

$Hf(\vec{x})$ $\forall \vec{x} \in \mathbb{R}^2$ is positive semidefinite because $e^{x_1-x_2} + e^{x_2-x_1} > 0$ and $\det(Hf(\vec{x})) = 0$

$\therefore$ $\vec{x}^*$ are global minimizers.

7. $\begin{pmatrix} a_1 & a_2 & a_3 \\ 2 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}$ Conditions:

$a_1 > 0$

$a_1 - 2a_2 > 0 \Rightarrow a_1 > 2a_2$

$a_1 \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} - a_2 \begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} + a_3 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 0$

$2a_1 - 4a_2 - a_3 = 0$

If $a_1 = 4$, then

$4 > 2a_2$ or $2 > a_2$; let $a_2 = 1$.

Then

$2(4) - 4(1) - a_3 = 0 \Rightarrow a_3 = 4$

8. A function with no critical points in $\mathbb{R}^3$ is, for example,

$$f(\vec{x}) = x_1 + x_2 + x_3$$

because

$$\nabla f(\vec{x}) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$ cannot be zero.

9. If $A = B^T B$, then $Q_A(\vec{x})$ (the quadratic form associated with A) is

$$Q_A(x) = x^T B^T B x = (Bx)^T (Bx) = \|Bx\|^2 \geq 0$$

$\therefore$ A is positive semidefinite.

10.
```
function [A] = randomPSD(n)
    B = rand(n,n);
    A = B' * B;

end

function [z] = quadraticForm(A, x1, x2)
    z = A(1,1) * x1.^2 + (A(1,2) + A(2,1)) * x1.*x2 +
        A(2,2) * x2.^2;
end
```

```
>> [x1, x2] = meshgrid(-2:0.2:2);
z = quadraticForm(randomPSD(2), x1, x2);
mesh(x1, x2, z);
```