# HW # 4 Math 529 - Spring 14

Solutions

1. Abstract on bordered Hessians. No specific correct answer exists.

2. $L(x,y,\lambda) = xy + \lambda(2 - x - 2y)$

$$\nabla L(x,y,\lambda) = \begin{pmatrix} y - \lambda \\ x - 2\lambda \\ 2 - x - 2y \end{pmatrix} = \vec{0} \Rightarrow \left.\begin{array}{ll} y - \lambda = 0 & (1) \\ x - 2\lambda = 0 & (2) \\ 2 - x - 2y = 0 & (3) \end{array}\right\}$$

From (1) & (2) in (3)

$y = \lambda$ ; $x = 2\lambda$

$2 - (2\lambda) - 2(\lambda) = 0$

$2 - 2\lambda - 2\lambda = 0$

$2 - 4\lambda = 0 \Rightarrow \lambda = \frac{2}{4} = \frac{1}{2}$ $\therefore$ $\boxed{x = 1 \text{ \& } y = \frac{1}{2}}$

Bordered Hessian:

$$\bar{H} = \begin{vmatrix} 0 & -1 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{vmatrix} = -(-1)\begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix} - 2\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} = 2 + 2 > 0$$

$\therefore$ $(1, \frac{1}{2})$ is a local maximizer.

3. $L(x,y,\lambda) = xy + \lambda(4 - x^2 - y^2)$

$$\nabla L(x,y,\lambda) = \begin{pmatrix} y - 2x\lambda \\ x - 2y\lambda \\ 4 - x^2 - y^2 \end{pmatrix} = \vec{0} \Rightarrow \left.\begin{array}{ll} y - 2x\lambda = 0 & (1) \\ x - 2y\lambda = 0 & (2) \\ 4 - x^2 - y^2 = 0 & (3) \end{array}\right\}$$

(1) in (2):

$y = 2x\lambda$

$x - 2(2x\lambda)\lambda = 0$

$x - 4x\lambda^2 = 0$

$x(1 - 4\lambda^2) = 0 \Rightarrow x = 0$ or $1 - 4\lambda^2 = 0$.

If $x = 0$, from (1): $y = 0$; however (3) would be violated. $\therefore$ $x \neq 0$.

If $1 - 4\lambda^2 = 0 \Rightarrow \lambda^2 = \frac{1}{4} \Rightarrow \lambda = \pm\frac{1}{2}$

$\lambda = \frac{1}{2}$:

$y - x = 0 \Rightarrow \underline{y = x}$

So in (3):

$4 - 2x^2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Points:

$(\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2})$

$\lambda = -\frac{1}{2}$:

$y + x = 0 \Rightarrow y = -x$

From (3) again $x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

So $y = \mp\sqrt{2}$

Points:

$(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

$$\bar{H} = \begin{vmatrix} 0 & -2x & -2y \\ -2x & -2\lambda & 1 \\ -2y & 1 & -2\lambda \end{vmatrix} = -(-2x)\begin{pmatrix} -2x & 1 \\ -2y & -2\lambda \end{pmatrix} - 2y\begin{pmatrix} -2x & -2\lambda \\ -2y & 1 \end{pmatrix}$$

$$= 2x\left(4x\lambda + 2y\right) - 2y\left(-2x - 4y\lambda\right)$$

$$= 8x^2\lambda + 4xy + 4xy + 8y^2\lambda$$

$$= 8x^2\lambda + 8xy + 8y^2\lambda$$

For $(\sqrt{2}, \sqrt{2})$, $\lambda = \frac{1}{2}$:

$\bar{H} = 8(2)\left(\frac{1}{2}\right) + 8(\sqrt{2})(\sqrt{2}) + 8(2)\frac{1}{2} = 32 > 0$, so $(\sqrt{2}, \sqrt{2})$ is a local maximizer

For $(-\sqrt{2}, -\sqrt{2})$, $\lambda = \frac{1}{2}$:

$\bar{H} = 8(2)\left(\frac{1}{2}\right) + 8(-\sqrt{2})(-\sqrt{2}) + 8(2)\left(\frac{1}{2}\right) = 32 > 0$, so $(-\sqrt{2}, -\sqrt{2})$ is a local maximizer

For $(\sqrt{2}, -\sqrt{2})$, $\lambda = -\frac{1}{2}$

$\bar{H} = 8(2)\left(-\frac{1}{2}\right) + 8(\sqrt{2})(-\sqrt{2}) + 8(2)\left(-\frac{1}{2}\right) = -32 < 0$, so $(\sqrt{2}, -\sqrt{2})$ is a local minimizer.

For $(-\sqrt{2}, \sqrt{2})$, $\lambda = -\frac{1}{2}$

$\bar{H} = 8(2)\left(-\frac{1}{2}\right) + 8(-\sqrt{2})(\sqrt{2}) + 8(2)\left(-\frac{1}{2}\right) = -32 < 0$, so $(-\sqrt{2}, \sqrt{2})$ is a local minimizer.

4. $L(x,y,\lambda_1,\lambda_2) = x+y+\lambda_1(4-x^2-y^2)+\lambda_2(0-x+y)$

$$\nabla L(x,y,\lambda_1,\lambda_2) = \begin{pmatrix} 1-2\lambda_1 x-\lambda_2 \\ 1-2\lambda_1 y+\lambda_2 \\ 4-x^2-y^2 \\ y-x \end{pmatrix} = \vec{0}$$

From (3) & (4):

$y-x=0 \Rightarrow y=x$

$4-2x^2=0 \Rightarrow x^2=2 \Rightarrow x=\pm\sqrt{2}$ & $y=\pm\sqrt{2}$

In (1) & (2):

$$1-2\lambda_1(\pm\sqrt{2})-\lambda_2=0$$
$$1-2\lambda_1(\pm\sqrt{2})+\lambda_2=0$$
$$\overline{\phantom{xxxxxxxxxxxxxxxx}}$$
$$2-4\lambda_1(\pm\sqrt{2})=0 \Rightarrow \lambda_1=\frac{\pm\sqrt{2}}{4} \quad ; \quad \lambda_2=0$$

$$\bar{H}=\begin{vmatrix} 0 & 0 & -2x & -2y \\ 0 & 0 & -1 & 1 \\ -2x & -1 & -2\lambda_1 & 0 \\ -2y & 1 & 0 & -2\lambda_1 \end{vmatrix}$$

This bordered Hessian does <u>not</u> provide information about the nature of the points $(\sqrt{2},\sqrt{2})$, $(-\sqrt{2},-\sqrt{2})$. The reason is that the problem is overconstrained in the sense that the second constraint: $x-y=0$ does not contribute to the localization of the solutions. In fact, $(\sqrt{2},\sqrt{2})$ & $(-\sqrt{2},-\sqrt{2})$ would still be solutions $\rightarrow$

without this constraint. Note also that $\lambda_2 = 0$ which further confirms the redundancy of the constraint.

If we remove the second constraint we have:

$$\bar{H} = \begin{vmatrix} 0 & -2x & -2y \\ -2x & -2\lambda_1 & 0 \\ -2y & 0 & -2\lambda_1 \end{vmatrix}$$

@ $(\sqrt{2}, \sqrt{2})$:

$$\bar{H} = \begin{vmatrix} 0 & -2\sqrt{2} & -2\sqrt{2} \\ -2\sqrt{2} & -\frac{\sqrt{2}}{2} & 0 \\ -2\sqrt{2} & 0 & -\frac{\sqrt{2}}{2} \end{vmatrix} = 2\sqrt{2}\begin{vmatrix} -2\sqrt{2} & 0 \\ -2\sqrt{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} - 2\sqrt{2}\begin{vmatrix} -2\sqrt{2} & -\frac{\sqrt{2}}{2} \\ -2\sqrt{2} & 0 \end{vmatrix}$$

$$= 2\sqrt{2}(2 - 0) - 2\sqrt{2}(0 - 2) = 8\sqrt{2} > 0 \;\therefore$$ $(\sqrt{2}, \sqrt{2})$ is a local maximizer

@ $(-\sqrt{2}, -\sqrt{2})$:

$$\bar{H} = \begin{vmatrix} 0 & 2\sqrt{2} & 2\sqrt{2} \\ 2\sqrt{2} & \frac{\sqrt{2}}{2} & 0 \\ 2\sqrt{2} & 0 & \frac{\sqrt{2}}{2} \end{vmatrix} = -2\sqrt{2}\begin{vmatrix} 2\sqrt{2} & 0 \\ 2\sqrt{2} & \frac{\sqrt{2}}{2} \end{vmatrix} + 2\sqrt{2}\begin{vmatrix} 2\sqrt{2} & \frac{\sqrt{2}}{2} \\ 2\sqrt{2} & 0 \end{vmatrix}$$

$$= -2\sqrt{2}(2) + 2\sqrt{2}(-2) = -8\sqrt{2} < 0 \;\therefore$$ $(-\sqrt{2}, -\sqrt{2})$ is a local minimizer

5. $L(x,y,z,\lambda_1,\lambda_2) = x^2+y^2-z^2+\lambda_1(2-x-y+z)+\lambda_2(0-x-z)$

$$\nabla_{x,y,z} L(x,y,z,\lambda_1,\lambda_2) = \begin{pmatrix} 2x-\lambda_1-\lambda_2 \\ 2y-\lambda_1 \\ 2z+\lambda_1-\lambda_2 \end{pmatrix} = \vec{0}$$

$x+y-z \geq 2$

$x+z \geq 0$

$\lambda_1(2-x-y+z)=0$

$\lambda_2(-x-z)=0$

$\lambda_1, \lambda_2 \geq 0$

6. $L(x,\lambda) = 3x_1 + x_1^3 + 2x_2 + \frac{1}{3}x_2^3 + \lambda_1(0+x_4-x_3+1) +$
$\lambda_2(0+x_3-x_4+1) + \lambda_3(0-100\sin(-x_3-1)-100\sin(-x_4-1)$
$-50+x_1) + \lambda_4(0-100\sin(x_3-1)-100\sin(x_3-x_4-1)-50$
$+x_2) + \lambda_5(0-x_1) + \lambda_6(0-x_2) + \lambda_7(0-x_3) + \lambda_8(0-x_4)$

$$\nabla_x L(x,\lambda) = \begin{pmatrix} 3+3x_1^2+\lambda_3-\lambda_5 \\ 2+x_2^2+\lambda_4-\lambda_6 \\ -\lambda_1+\lambda_2+100\lambda_3\cos(-x_3-1)-100\lambda_4\cos(x_3-1)- \\ 100\lambda_4\cos(x_3-x_4-1)-\lambda_7 \\ \lambda_1-\lambda_2+100\lambda_3\cos(-x_4-1)+100\lambda_4\cos(x_3-x_4-1)-\lambda_8 \end{pmatrix} = \vec{0}$$

$-x_4 + x_3 - 1 \geq 0$

$-x_3 + x_4 - 1 \geq 0$

$100 \sin(-x_3 - 1) + 100 \sin(-x_4 - 1) + 50 - x_1 = 0$

$100 \sin(x_3 - 1) + 100 \sin(x_3 - x_4 - 1) + 50 - x_2 = 0$

$x_1, x_2, x_3, x_4 \geq 0$

$\lambda_1(-x_4 + x_3 - 1) = 0$

$\lambda_2(-x_3 + x_4 - 1) = 0$

$\lambda_1, \lambda_2 \geq 0$

7. $\min f(x,y) = xy - x^2 - y^2$

s.t. $-x^2 - y^2 \geq -2$

$x + y \geq 0$

$$L(x,y,\lambda_1,\lambda_2) = xy - x^2 - y^2 + \lambda_1(-2 + x^2 + y^2) + \lambda_2(0 - x - y)$$

$$\nabla_{x,y} L(x,y,\lambda_1,\lambda_2) = \begin{pmatrix} y - 2x + 2\lambda_1 x - \lambda_2 \\ x - 2y + 2\lambda_1 y - \lambda_2 \end{pmatrix} = \vec{0}$$

$-x^2 - y^2 \geq -2$

$x + y \geq 0$

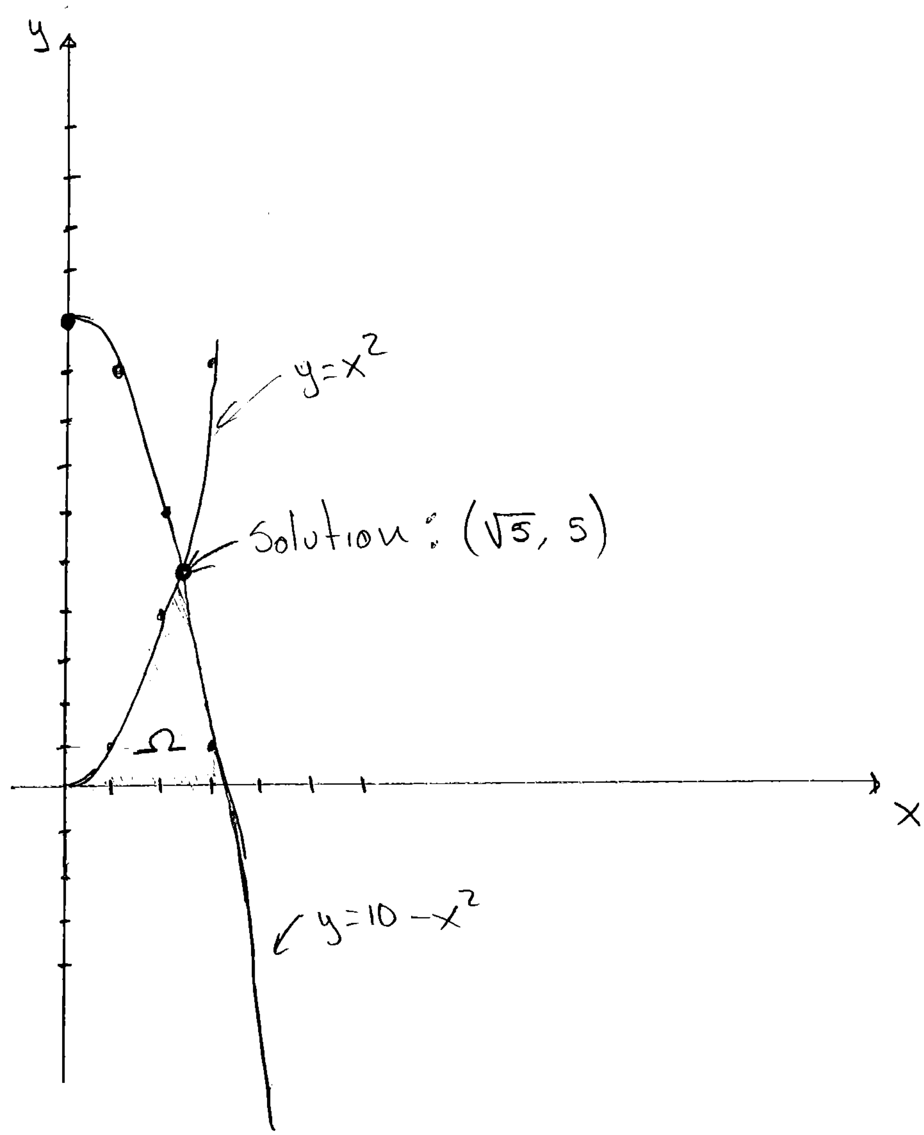
$\lambda_1(-2+x^2+y^2)=0$

$\lambda_2(-x-y)=0$

$\lambda_1, \lambda_2 \geq 0$

$\longrightarrow$ See addendum for the rest of the solution to this problem.

8.



$$L(x,y,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = x+y+\lambda_1(x^2-y)+\lambda_2(10-x^2-y)+ \lambda_3(-x)+\lambda_4(-y)$$

$$\nabla_{x,y} L(x,y,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \begin{pmatrix} 1+2\lambda_1 x - 2\lambda_2 x - \lambda_3 \\ \\ 1-\lambda_1-\lambda_2-\lambda_4 \end{pmatrix} \begin{matrix} (1) \\ \\ (2) \end{matrix} = \vec{0}$$

$x^2 - y \geq 0$ (3)

$10 - x^2 - y \geq 0$ (4)

$x, y \geq 0$ (5)

$\lambda_1(x^2-y)=0$ (6)

$\lambda_2(10-x^2-y)=0$ (7)

$\lambda_3(x)=0$ (8)

$\lambda_4(y)=0$ (9)

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ (10)

@ $(\sqrt{5}, 5)$:

(3), (4), (5), (6), (7) are clearly satisfied.

From (8): $\lambda_3(\sqrt{5})=0 \Rightarrow \lambda_3=0$

From (9): $\lambda_4(5)=0 \Rightarrow \lambda_4=0$

Substituting $x, \lambda_3, \lambda_4$ in (1) & (2):

$$\left.\begin{aligned} &1+2\sqrt{5}\lambda_1 - 2\sqrt{5}\lambda_2 = 0 \\ &1-\lambda_1-\lambda_2 = 0 \end{aligned}\right\} \Rightarrow \lambda_1 = 1-\lambda_2$$

$$1+2\sqrt{5}(1-\lambda_2) - 2\sqrt{5}\lambda_2 = 0$$

$$1+2\sqrt{5} - 2\sqrt{5}\lambda_2 - 2\sqrt{5}\lambda_2 = 0$$

$$1+2\sqrt{5} - 4\sqrt{5}\lambda_2 = 0 \Rightarrow \lambda_2 = \frac{1+2\sqrt{5}}{4\sqrt{5}} > 0$$

$$\lambda_1 = 1 - \frac{1+2\sqrt{5}}{4\sqrt{5}} = \frac{4\sqrt{5}-1-2\sqrt{5}}{4\sqrt{5}} = \frac{2\sqrt{5}-1}{4\sqrt{5}} > 0$$

So (1), (2) & (10) are also satisfied.

LICQ: Active constraints: $\underbrace{x^2-y \geq 0}_{c_1}$, $\underbrace{10-x^2-y \geq 0}_{c_2}$

$$\left.\begin{aligned} &\nabla c_1(x,y) = \begin{pmatrix} 2x \\ -1 \end{pmatrix} \quad @(\sqrt{5},5): \nabla c_1(\sqrt{5},5) = \begin{pmatrix} 2\sqrt{5} \\ -1 \end{pmatrix} \\ &\nabla c_2(x,y) = \begin{pmatrix} -2x \\ -1 \end{pmatrix} \quad @(\sqrt{5},5): \nabla c_2(\sqrt{5},5) = \begin{pmatrix} -2\sqrt{5} \\ -1 \end{pmatrix} \end{aligned}\right\}$$

These vectors are linearly independent so LICQ holds at $(\sqrt{5}, 5)$

MFCQ: Is there $w$, such that $w^T \nabla c_1 > 0$ & $w^T \nabla c_2 > 0$?

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Rightarrow \left.\begin{aligned} &w^T \nabla c_1(\sqrt{5},5) = 2\sqrt{5}w_1 - w_2 > 0 \\ &w^T \nabla c_2(\sqrt{5},5) = -2\sqrt{5}w_1 - w_2 > 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &w_1 = 0 \\ &\& \\ &w_2 < 0 \end{aligned}$$

So, e.g., $w = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ satisfies MFCQ

$x^2 - y \geq 0 \quad (3)$

$-x^2 - y + 10 \geq 0 \quad (4)$

$x, y \geq 0 \quad (5)$

$\lambda_1 (x^2 - y) = 0 \quad (6)$

$\lambda_2 (-x^2 - y + 10) = 0 \quad (7)$

$\lambda_3 x = 0 \quad (8)$

$\lambda_4 y = 0 \quad (9)$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \quad (10)$

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@ $(0,0)$

(3), (4), (5), (6), (8) & (9) are satisfied

From (7) $\lambda_2 = 0 \geq 0$

(1) $\lambda_3 = 1 \geq 0$

Thus, from (2): The KKT conditions hold as long as $\lambda_1 - \lambda_4 = 0 \Rightarrow \lambda_1 = \lambda_4 \geq 0$

9. The feasible set is the same as in problem 8. The solution now is the point $(0,0)$.

At this point, the active constraints are: $x^2-y \geq 0$, $x \geq 0$ and $y \geq 0$. Therefore, for LICQ we have:

$$\nabla c_1(x,y) = \begin{pmatrix} -2x \\ 1 \end{pmatrix} \qquad \nabla c_3(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \nabla c_4(x,y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\nabla c_1(0,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \nabla c_3(0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \nabla c_4(0,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Since there are three active constraints, but the problem has two dimensions, LICQ clearly does not hold.

For MFCQ, if $w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then $w^T \nabla c_1(0,0) > 0$ &
$w^T \nabla c_3(0,0) > 0$ &
$w^T \nabla c_4(0,0) > 0$

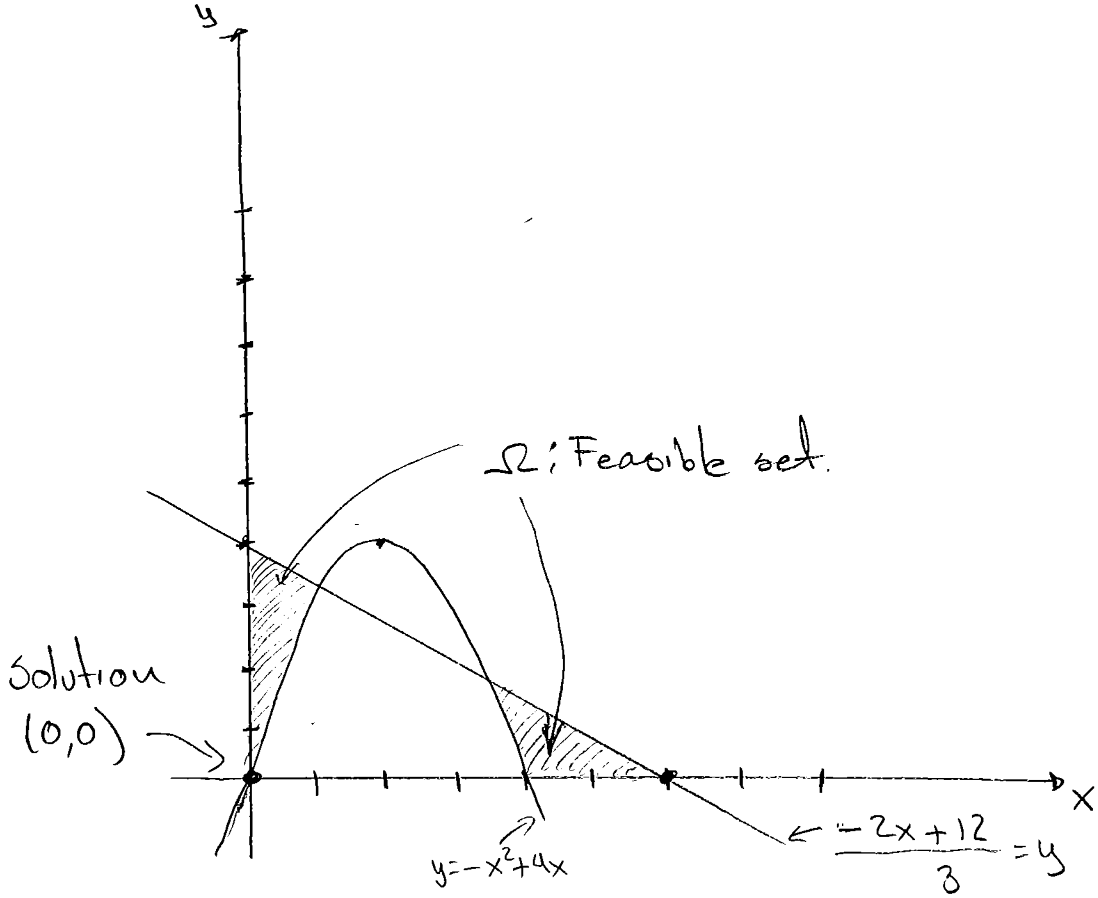
$\therefore$ MFCQ holds.

Checking KKT conditions:

$$L(x,y,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = x - \lambda_1(x^2-y) - \lambda_2(-x^2-y+10) - \lambda_3 x - \lambda_4 y$$

$$\nabla_{x,y} L(x,y,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \begin{pmatrix} 1 - 2\lambda_1 x + 2\lambda_2 x - \lambda_3 \\ \lambda_1 + \lambda_2 - \lambda_4 \end{pmatrix} = \vec{0} \quad \begin{matrix} \leftarrow (1) \\ \leftarrow (2) \end{matrix}$$

10.




At the solution point, there are three active constraints. By the same reasoning as before, i.e. more constraints than decision variables, LICQ does not hold.

For MFCQ:

$$\nabla c_1(x,y) = \begin{pmatrix} -2x+4 \\ -1 \end{pmatrix} \quad @\ (0,0) \quad \nabla c_1(0,0) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\nabla c_3(x,y) = \nabla c_3(0,0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\nabla c_4(x,y) = \nabla c_4(0,0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

A vector that satisfies $w^T \nabla c_i(0,0) > 0$, $i = \{1,3,4\}$ should satisfy: $4w_1 > w_2$, $w_1 < 0$ & $w_2 < 0$.
One such vector is $\left(-\frac{1}{8}, -1\right)$. So MFCQ is satisfied.

KKT conditions:

$$L(x,y,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = 2x_1 + x_2 + \lambda_1(0 - x^2 + 4x - y) + \lambda_2(-12 + 2x + 3y) + \lambda_3(-x) + \lambda_4(-y)$$

$$\nabla_{x,y} L(x,y,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \begin{pmatrix} 2 - 2\lambda_1 x + 4\lambda_1 + 2\lambda_2 - \lambda_3 \\ 1 - \lambda_1 + 3\lambda_2 - \lambda_4 \end{pmatrix} = 0 \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

$x^2 - 4x + y \geq 0$ (3)

$-2x - 3y \geq -12$ (4)

$x, y \geq 0$ (5)

$\lambda_1(x^2 - 4x + y) = 0$ (6)

$\lambda_2(-12 + 2x + 3y) = 0$ (7)

$\lambda_3 x = 0$ (8)

$\lambda_4 y = 0$ (9)

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ (10)

@ (0,0)

From (7): $\lambda_2 = 0$

In (1) & (2):

$$2 + 4\lambda_1 - \lambda_3 = 0 \qquad (1')$$

$$1 - \lambda_1 - \lambda_4 = 0 \qquad (2')$$

Exploring cases:

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | Conclusion |
|---|---|---|---|
| 0 | 0 | 0 | violates 1' & 2' |
| 0 | 0 | >0 | violates 1' |
| 0 | >0 | 0 | violates 2' |
| 0 | >0 | >0 | KKT conditions are valid with $x = y = 0$, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 2$, $\lambda_4 = 1$ |

Addendum problem #7:

Solving KKT system:

Case 1: $\lambda_1 = 0$, $\lambda_2 = 0$

$y - 2x = 0 \Rightarrow y = 2x$

$x - 2y = 0 \Rightarrow x - 2(2x) = 0$

$x - 4x = 0$

$-3x = 0 \Rightarrow x = 0$

$\therefore y = 0$

(0,0) with $\lambda_1 = \lambda_2 = 0$ satisfies the KKT conditions.

Case 2: $\lambda_1 = 0$, $\lambda_2 > 0$

$$\left.\begin{array}{l} y - 2x - \lambda_2 = 0 \\ x - 2y - \lambda_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \lambda_2 = \lambda_2 \\ y - 2x = x - 2y \\ 3y = 3x \Rightarrow \underline{y = x} \end{array}$$

In $\lambda_2 \underbrace{(-x-y)}_{0} = 0$

$-x - x = 0 \therefore x = 0 = y$

But this means that $\lambda_2 = y - 2x = 0$ which contradicts our assumption $\lambda_2 > 0$.

Case 3: $\lambda_1 > 0 \quad \lambda_2 = 0$

$$y - 2x + 2\lambda_1 x = 0 \Rightarrow y = 2x - 2\lambda_1 x$$

$$x - 2y + 2\lambda_1 y = 0 \Rightarrow x - 2(2x - 2\lambda_1 x) + 2\lambda_1(2x - 2\lambda_1 x) = 0$$

$$x - 4x + 4\lambda_1 x + 4\lambda_1 x - 4\lambda_1^2 x = 0$$

$$x - 4x + 8\lambda_1 x - 4\lambda_1^2 x = 0$$

$$x(1 - 4 + 8\lambda_1 - 4\lambda_1^2) = 0$$

$\therefore \quad x = 0$ or

$$-3 + 8\lambda_1 - 4\lambda_1^2 = 0$$

* $x$ cannot be zero as that would make $y$ zero, making $\lambda_1 = \dfrac{2x - y}{2x}$ undefined.

* Solving $-3 + 8\lambda_1 - 4\lambda_1^2$ we obtain $\lambda_1 = \frac{1}{2}$ or $\lambda_1 = \frac{3}{2}$

If $\lambda_1 = \frac{1}{2}$

$$y = 2x - 2\left(\tfrac{1}{2}\right)x = 2x - x = x$$

In $\lambda_1(-2 + x^2 + y^2) = 0$:

$$-2 + 2x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

$$y = \pm 1$$

| | |
|---|---|
| $(1,1)$ with $\lambda_1 = \frac{1}{2}$ satisfies the KKT conditions | $(-1,-1)$ with $\lambda_1 = \frac{1}{2}$ violates $x + y \geq 0$ |

If $\lambda_1 = \frac{3}{2}$

$$y = 2x - 2\left(\frac{3}{2}\right)x = 2x - 3x = -x$$

In $\lambda_1(-2 + x^2 + y^2) = 0$:

$$-2 + x^2 + (-x)^2 = -2 + 2x^2 \Rightarrow x = \pm 1$$
$$y = \mp 1$$

$(1, -1)$ with $\lambda_1 = \frac{3}{2}$ satisfies the KKT conditions

$(-1, 1)$ with $\lambda_1 = \frac{3}{2}$ satisfies the KKT conditions.

Case 4: $\lambda_1 > 0 \quad \lambda_2 > 0$

$$-2 + x^2 + y^2 = 0 \qquad \Rightarrow -2 + (-y)^2 + y^2 = 0$$
$$-x - y = 0 \Rightarrow x = -y \qquad -2 + 2y^2 = 0 \Rightarrow y = \pm 1$$
$$x = \mp 1$$

Since these points are the same we found in the previous case, we can stop our search.

The extrema are:

$f(0,0) = 0$

$f(1,1) = -1$

$f(1,-1) = -3$

$f(-1,1) = -3$

$\therefore$ $(1,-1)$ & $(-1,1)$ are the minimizers