Math 529 - Fundamentals of optimization

HW5

1. $Q = -xy + \frac{\mu}{2}(x+2y-4)^2$

$$\nabla Q = \begin{pmatrix} -y + \mu(x+2y-4) \\ -x + 2\mu(x+2y-4) \end{pmatrix}$$

If a critical point of Q is a minimizer, then its Hessian at that point (at least) must be positive (semi) definite. So

$$HQ = \begin{pmatrix} \mu & -1+2\mu \\ -1+2\mu & 4\mu \end{pmatrix}$$

For HQ to be positive definite, we need

$\mu > 0$ and $4\mu^2 - (2\mu-1)^2 > 0$

$$4\mu^2 > (2\mu-1)^2 = 4\mu^2 - 4\mu + 1$$

$$4\mu > 1$$

$$\underline{\mu > \frac{1}{4}}$$

2. $\min -\frac{1}{2}x^2$ subject to $x=1$.

a)
$$L(x,\lambda) = -\frac{1}{2}x^2 + \lambda(1-x)$$

$$\nabla L(x,\lambda) = \begin{pmatrix} -x-\lambda \\ 1-x \end{pmatrix} = \vec{0} \Rightarrow \begin{array}{l} x=1 \\ \lambda = -x = -1 \end{array}$$

(Since the problem has an equality constraint, the actual sign of λ is irrelevant.)

b)
$$AL(x,\lambda,\mu) = -\frac{1}{2}x^2 + \lambda(1-x) + \frac{\mu}{2}(1-x)^2$$

$$\nabla_x AL(x,\lambda,\mu) = -x - \lambda + \mu(1-x)(-1)$$

$$\nabla_x^2 AL(x,\lambda,\mu) = -1+\mu$$

If AL is convex, then $\nabla_x^2 AL(x,\lambda,\mu) > 0$, which means

$$-1+\mu > 0 \Rightarrow \underline{\mu > 1}$$

If $-1+\mu < 0 \Rightarrow \mu < 1$, then the augmented Lagrangian is concave and therefore it is unbounded below

If $\mu = 1$, the augmented Lagrangian is both convex and concave, which means that it is a line.

4.

Tableau 1:

p	x	y	z	s_1	s_2	s_3	C
1	-5	4	-3	0	0	0	0
0	5	0	5	1	0	0	100
0	0	5	-5	0	1	0	50
0	5	-5	0	0	0	1	50

Pivot column: x, Pivot element 5 (4 th row)

Actions: $R_1 + R_4 \to R_1$, $-R_2 + R_4 \to R_2$, $R_4/5 \to R_4$.

Tableau 2:

p	x	y	z	s_1	s_2	s_3	C
1	0	-1	-3	0	0	1	50
0	0	-5	-5	-1	0	1	-50
0	0	5	-5	0	1	0	50
0	1	-1	0	0	0	$1/5$	10

Actions: $-R_2 \to R_2$ (correct sign of s_1 column)

p	x	y	z	s_1	s_2	s_3	C
1	0	-1	-3	0	0	1	50
0	0	5	5	1	0	-1	50
0	0	5	-5	0	1	0	50
0	1	-1	0	0	0	$1/5$	10

Pivot column: z, Pivot element 5 (2nd row)

Actions: $R_2 + R_3 \to R_3$, $3R_2 + 5R_1 \to R_1$, $R_2/5 \to R_2$

Tableau 3:

p	x	y	z	s_1	s_2	s_3	C
5	0	10	0	3	0	2	400
0	0	1	1	$1/5$	0	$-1/5$	10
0	0	10	0	1	1	-1	100
0	1	-1	0	0	0	$1/5$	10

Actions: $R_1/5 \to R_1$ (correct magnitude of p)

p	x	y	z	s_1	s_2	s_3	C
1	0	2	0	$3/5$	0	$2/5$	80
0	0	1	1	$1/5$	0	$-1/5$	10
0	0	10	0	1	1	-1	100
0	1	-1	0	0	0	$1/5$	10

Optimal solution $p = 80$

$x = 10$

$y = 0$

$z = 10$

5.

$$\max\ 10x + 4y + 8z$$

subject to

$$2x + y + 3z \leq 1000$$

$$x + y + z \leq 650$$

$$x, y, z \geq 0$$

Solution: 5000 dollars with 500 calculus textbooks. Sorry humanities & management majors!!

6. Dual of

$$\max \quad x+y+z+w$$

$$\text{subject to} \quad x+y+z \leq 3,$$
$$y+z+w \leq 4,$$
$$x+z+w \leq 5,$$
$$x+y+w \leq 6,$$
$$x,y,z,w \geq 0$$

which can be written as

$$\max \quad c^T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

subject to

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \leq \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$$

$$x,y,z,w \geq 0$$

So the dual is:

$$\min \quad \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}^T \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = 3\lambda_1 + 4\lambda_2 + 5\lambda_3 + 6\lambda_4$$

subject to

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

Or

$$\min \quad 3\lambda_1 + 4\lambda_2 + 5\lambda_3 + 6\lambda_4$$

subject to

$$\lambda_1 + \lambda_3 + \lambda_4 \geq 1$$

$$\lambda_1 + \lambda_2 + \lambda_4 \geq 1$$

$$\lambda_1 + \lambda_2 + \lambda_3 \geq 1$$

$$\lambda_2 + \lambda_3 + \lambda_4 \geq 1$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

7. The original payoff matrix is

$$\begin{array}{cc} & S \\ D & \begin{pmatrix} -200 & -300 & 300 \\ -500 & 500 & -100 \\ -500 & 0 & 0 \end{pmatrix} \end{array}$$

For D, choosing the 3rd option is never good because D would never gain votes. Similarly, for S, who wants to minimize D's gains, choosing the 3rd option leads to the least number of votes "stolen" from D. So, since the 3rd row and 3rd column would never be chosen, they can be eliminated from the matrix. Therefore the new payoff matrix is

$$\begin{array}{cc} & S \\ D & \begin{pmatrix} -200 & -300 \\ -500 & 500 \end{pmatrix} \end{array}$$

Shifting the payoffs; by adding 500:

$$\begin{array}{cc} & S \\ D & \begin{pmatrix} 300 & 200 \\ 0 & 1000 \end{pmatrix} \end{array}$$

D has to solve the problem

$$\max \quad U$$

$$\text{subject to} \quad 300x + 200y \geq U$$

$$1000y \geq U$$

$$x + y = 1$$

$$x, y \geq 0$$

If $\frac{x}{U} = w_1$, $\frac{y}{U} = w_2$:

The program can be written as:

$$\min \quad w_1 + w_2$$

$$\text{subject to} \quad 300w_1 + 200w_2 \geq 1$$

$$1000w_2 \geq 1$$

$$w_1, w_2 \geq 0$$

whose solution is

$$\frac{1}{U} = w_1^* + w_2^* = \frac{11}{3000} \quad , \quad w_1^* = \frac{1}{375} \quad , \quad w_2^* = \frac{1}{1000}$$

$$\therefore \quad x^* = w_1^* \left(\frac{3000}{11}\right) = \frac{8}{11}$$

$$y^* = w_2^* \left(\frac{3000}{11}\right) = \frac{3}{11}$$

The best outcome D can expect is

$U - 500 = \frac{3000}{11} - 500 = -227.27$ votes

j has to solve the problem

$$\min \ v$$

subject to
$$300x \leq v$$
$$200x + 1000y \leq v$$
$$x + y = 1$$
$$x, y \geq 0$$

Using $w_1 = \frac{x}{v}$ and $w_2 = \frac{y}{v}$, the program is

$$\max \ w_1 + w_2$$

subject to
$$300 w_1 \leq 1$$
$$200 w_1 + 1000 w_2 \leq 1$$
$$w_1, w_2 \geq 0$$

the solution of which is:

$\frac{1}{v} = w_1^* + w_2^* = \frac{11}{3000}$, with $w_1^* = \frac{1}{300}$, $w_2^* = \frac{1}{3000}$

which leads to the solution

$$\left.\begin{aligned} x^* &= w_1^* \cdot v = \frac{1}{300}\left(\frac{3000}{11}\right) = \frac{10}{11} \\ y^* &= w_2^* \cdot v = \frac{1}{3000}\left(\frac{3000}{11}\right) = \frac{1}{11} \end{aligned}\right\}$$ with an expected outcome of -227.27 votes.

Therefore,

Trick L. Down should spend $\frac{10}{11}$ of his time in Littleville and $\frac{1}{11}$ of his time in Metropolis.

Tax N. Spend should spend $\frac{8}{11}$ of his time in Littleville and $\frac{3}{11}$ of his time in Metropolis.