

# MATH529 – Fundamentals of Optimization

## Linear Programming and the Simplex Method

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## Linear Programming

### Definition

A linear program is one with linear objective function and linear constraints, which may include equality and inequality constraints.

Example:

$$\begin{aligned} \min \quad & -4x_1 + 2x_2 - 2x_3 + 13x_4, \\ \text{subject to} \quad & x_1 + x_2 \geq 3, \quad x_1 - x_3 + x_4 = 3, \quad x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

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## Linear Programming

### Appeal:

- It is often easier (and more appropriate) to model problems as linear programs
- KKT conditions are valid
- The feasible set is a *polytope* (a convex, connected, set with flat, polygonal faces).
- The solution is found at extreme points, thus reducing the search space to specific regions of the feasible set.
- A local optimum will also be a global optimum.

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## Linear Programming

### Appeal:

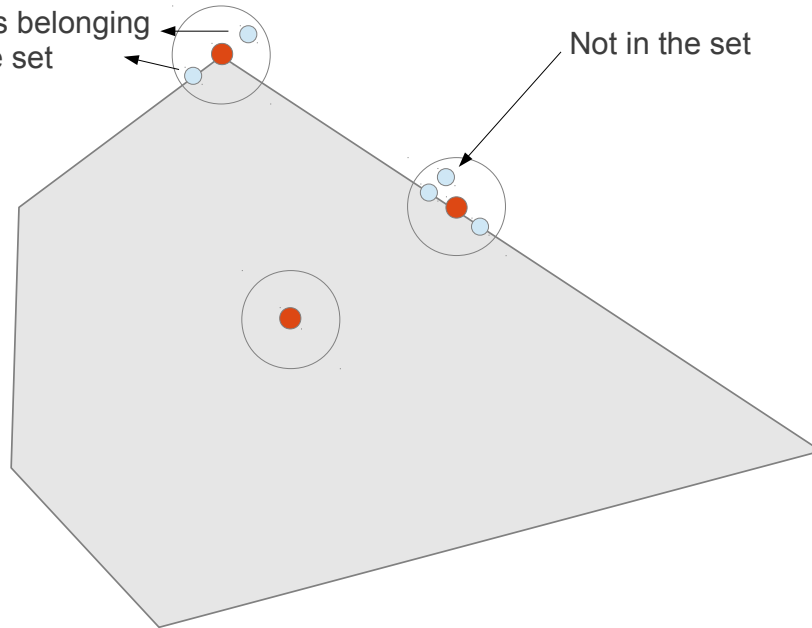
- It is often easier (and more appropriate) to model problems as linear programs
- KKT conditions are valid
- The feasible set is a *polytope* (a convex, connected, set with flat, polygonal faces).
- The solution is found at extreme points, thus reducing the search space to specific regions of the feasible set. **Although in some instances this idea is actually misleading.**
- A local optimum will also be a global optimum.

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## Linear Programming: Toward a solution method

Interior, boundary, extreme points:

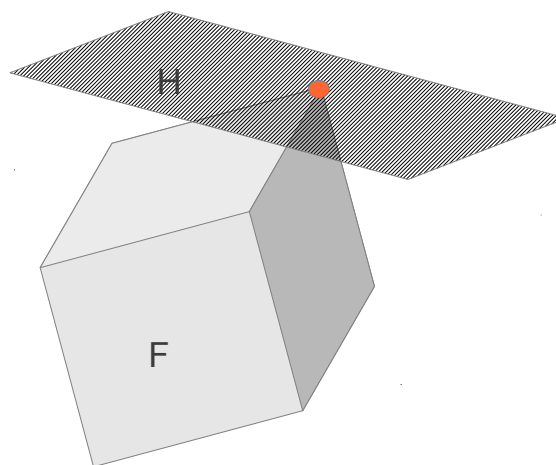
No line between  
points belonging  
to the set



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## Linear Programming: Toward a solution method

Supporting hyperplanes:



A supporting hyperplane  $H$  has one or more points in common with a convex set  $F$ , but  $F$  lies completely one side of  $H$ .

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## Linear Programming: Toward a solution method

- Theorem 1: Given  $\mathbf{u}$ , a boundary point of a closed convex set, there is at least one supporting hyperplane at  $\mathbf{u}$ .
- Theorem 2: For a closed convex set bounded from below, there is at least one extreme point in every supporting hyperplane.

Since in linear programming the objective function is linear, the hyperplane corresponding to the optimal value of the function will be a supporting hyperplane of the feasible set.

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## Linear Programming: Toward a solution method

- Theorem 1: Given  $\mathbf{u}$ , a boundary point of a closed convex set, there is at least one supporting hyperplane at  $\mathbf{u}$ .
- Theorem 2: For a closed convex set bounded from below, there is at least one extreme point in every supporting hyperplane.

Since in linear programming the objective function is linear, the hyperplane corresponding to the optimal value of the function will be a supporting hyperplane of the feasible set.

**We can therefore pay attention only to extreme points!**

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## Linear Programming: Toward a solution method

Example:

$$\max 40x + 30y$$

subject to:

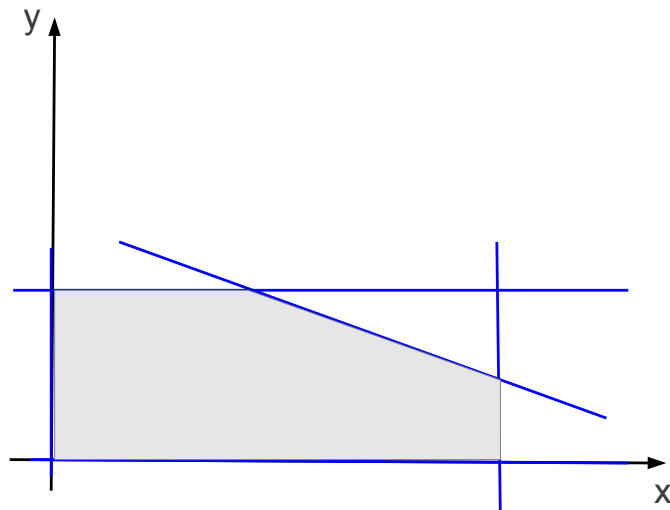
$$x + 2y \leq 24$$

$$0 \leq x \leq 16$$

$$0 \leq y \leq 8$$

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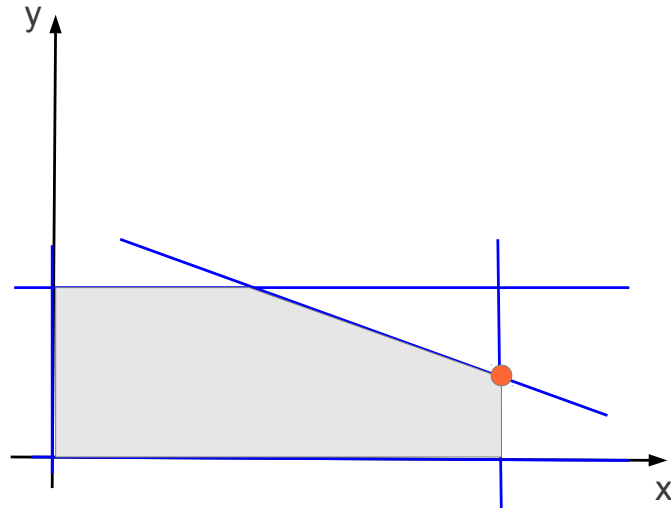
## Linear Programming: Toward a solution method



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## Linear Programming: Toward a solution method

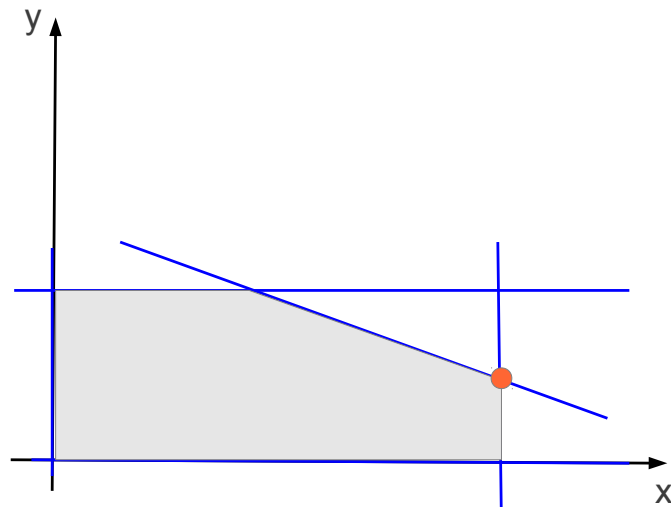
Dummy variables: Slacks and Surpluses:



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## Linear Programming: Toward a solution method

Dummy variables: Slacks and Surpluses:



**With the inexact satisfaction of a constraint, there is a slack or surplus related to one or more constraints.**

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## Linear Programming: Toward a solution method

Step 1: Transforming the program:

$$\max p = 40x + 30y + 0s_1 + 0s_2 + 0s_3 + 0s_4 + 0s_5$$

subject to:

$$x + 2y + s_1 = 24$$

$$x + s_2 = 16$$

$$y + s_3 = 8$$

$$x - s_4 = 0^*$$

$$y - s_5 = 0^*$$

$$x, y, s_1, s_2, s_3, s_4, s_5 \geq 0$$

\*For surpluses, we add  $-s_i$ , with  $s_i \geq 0$ .

## Linear Programming: Toward a solution method

Step 1: Transforming the program:

$$\max p = 40x + 30y$$

subject to:

$$x + 2y + s_1 = 24$$

$$x + s_2 = 16$$

$$y + s_3 = 8$$

$$x, y, s_1, s_2, s_3 \geq 0$$

## Linear Programming: Toward a solution method

Step 2: Generating extreme points:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

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## Linear Programming: Toward a solution method

Step 2: Generating extreme points:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

Let  $x = y = 0$ , then

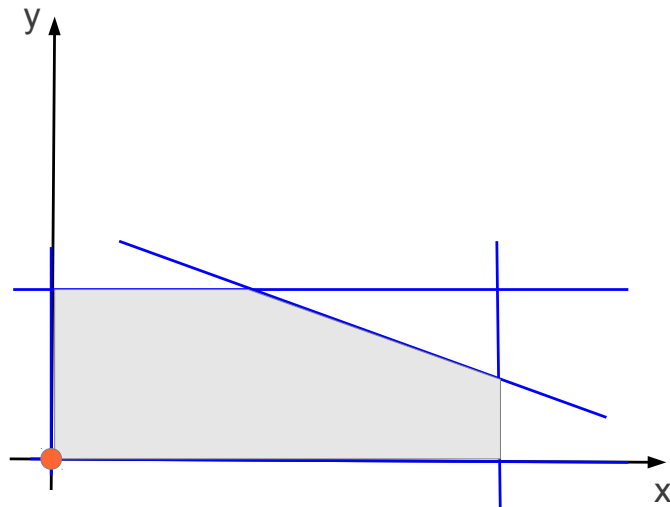
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

That is  $s_1 = 24$ ,  $s_2 = 16$ , and  $s_3 = 8$ .

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## Linear Programming: Toward a solution method



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## Linear Programming: Toward a solution method

Step 2: Generating extreme points:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

Let  $x = s_1 = 0$ , then

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

That is  $y = 12$ ,  $s_2 = 16$ , and  $s_3 = -4$ . (Violates nonnegativity restrictions!)

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## Linear Programming: Toward a solution method

Step 2: Generating extreme points:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

Let  $x = s_2 = 0$ , then

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ s_1 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

Invalid system!

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## Linear Programming: Toward a solution method

Step 2: Generating extreme points:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

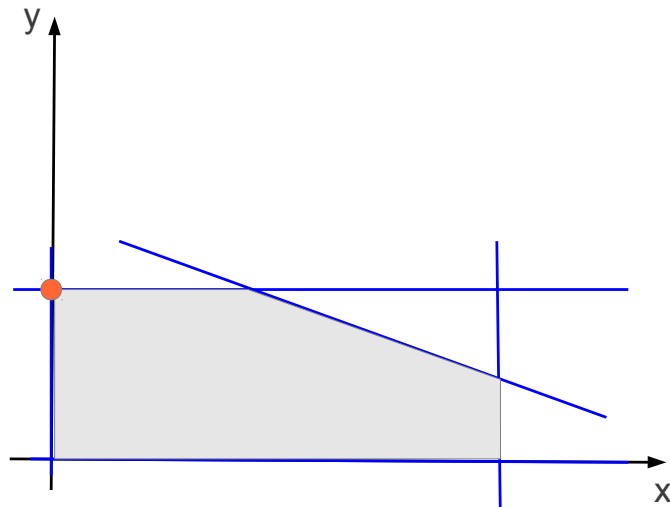
Let  $x = s_3 = 0$ , then

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

That is  $y = 8$ ,  $s_2 = 16$ , and  $s_3 = 8$ .

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## Linear Programming: Toward a solution method



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## Linear Programming: Toward a solution method

Step 2: Generating extreme points:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

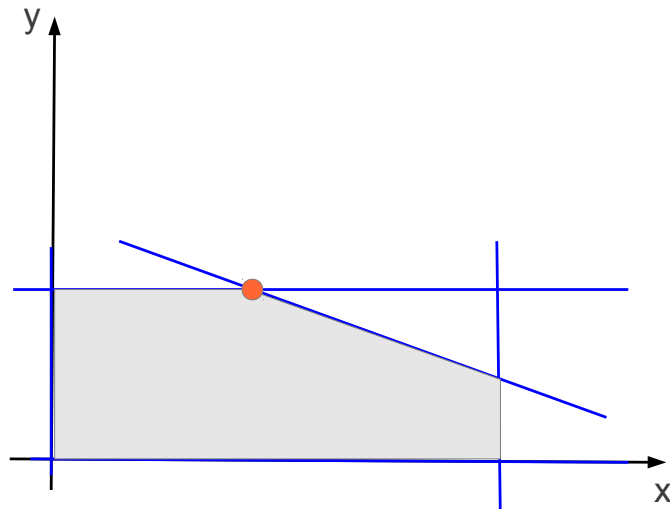
Let  $s_1 = s_3 = 0$ , then

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_2 \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 8 \end{bmatrix}$$

That is  $x = 8$ ,  $y = 8$ , and  $s_2 = 8$ .

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## Linear Programming: Toward a solution method



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## Linear Programming: Toward a solution method

We now have a method to systematically generate extreme points. But there are a few remaining questions:

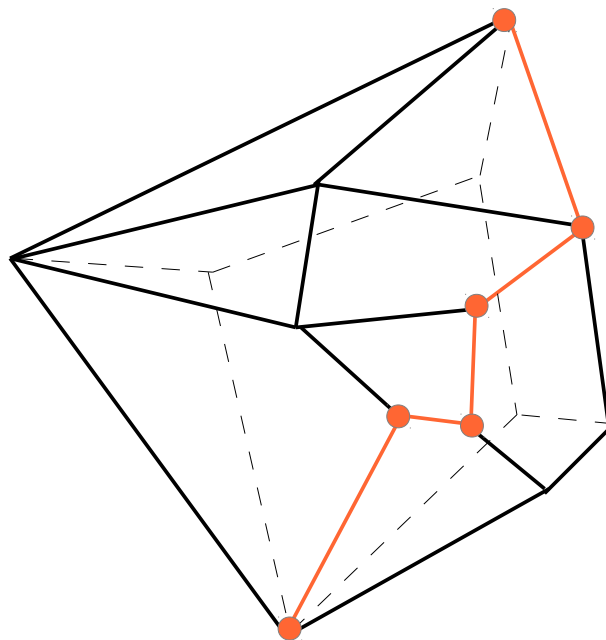
- How to select which columns to eliminate so that we generate as few extreme points as possible?
- How can we use the objective function to guide the search?
- How to avoid generating invalid moves?

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# The Simplex Method

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## The idea behind the Simplex Method



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## The Simplex Method

Simplex tableau:

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant
1	-40	-30	0	0	0	0
0	1	2	1	0	0	24
0	1	0	0	1	0	16
0	0	1	0	0	1	8

Basis at (0,0):

$$\begin{bmatrix} s_1 & s_2 & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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## The Simplex Method

Pivoting: Choose the pivot column, then the pivot element.

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant
1	-40	-30	0	0	0	0
0	1	2	1	0	0	24
0	1	0	0	1	0	16
0	0	1	0	0	1	8

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## The Simplex Method

Pivoting: Choose the pivot column, then the pivot element.

**Choose the column associated with the negative entry with the largest abs. value**

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant
1	<b>-40</b>	-30	0	0	0	0
0	<b>1</b>	2	1	0	0	24
0	<b>1</b>	0	0	1	0	16
0	<b>0</b>	1	0	0	1	8

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## The Simplex Method

Pivoting: Choose the pivot column, then the pivot element.

If we choose to replace  $s_1$ :

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant	
1	-40	-30	0	0	0	0	$40R_2 + R_1$
0	<b>1</b>	2	1	0	0	24	
0	1	0	0	1	0	16	$R_2 - R_3$
0	0	1	0	0	1	8	

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## The Simplex Method

Pivoting: Choose the pivot column, then the pivot element.

If we choose to replace  $s_1$ :

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant
1	0	50	40	0	0	960
0	<b>1</b>	2	1	0	0	24
0	0	2	1	-1	0	8
0	0	1	0	0	1	8

New basis:

$$\begin{bmatrix} x & s_2 & s_3 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus  $x = 24$ ,  $y = 0$ ,  $s_1 = 0$ ,  $s_2 = -8$ ,  $s_3 = 8$

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## The Simplex Method

Pivoting: Choose the pivot column, then the pivot element.

If we choose to replace  $s_1$ :

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant
1	0	50	40	0	0	960
0	<b>1</b>	2	1	0	0	24
0	0	2	1	-1	0	8
0	0	1	0	0	1	8

New basis:

$$\begin{bmatrix} x & s_2 & s_3 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus  $x = 24$ ,  $y = 0$ ,  $s_1 = 0$ ,  $s_2 = -8$ ,  $s_3 = 8$

**Invalid move!**

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## The Simplex Method

Pivoting: Choose the pivot column, then the pivot element.

If we choose to replace  $s_2$ :

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant	
1	-40	-30	0	0	0	0	$40R_3 + R_1$
0	1	2	1	0	0	24	$R_2 - R_3$
0	<b>1</b>	0	0	1	0	16	
0	0	1	0	0	1	8	

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## The Simplex Method

Pivoting: Choose the pivot column, then the pivot element.

If we choose to replace  $s_2$ :

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant
1	0	-30	0	40	0	640
0	0	2	1	-1	0	8
0	<b>1</b>	0	0	1	0	16
0	0	1	0	0	1	8

New basis:

$$\begin{bmatrix} s_1 & x & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus  $x = 16$ ,  $y = 0$ ,  $s_1 = 8$ ,  $s_2 = 0$ ,  $s_3 = 8$

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## The Simplex Method

Pivoting: Choose the pivot column, then the pivot element.

So how do we choose the pivot element?

Pick the positive elements in the pivot column,  
Divide the constant column by these elements,  
Select the element corresponding to the smallest quotient as pivot element

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## The Simplex Method

Pivoting: Choose the pivot column, then the pivot element.

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant	
1	-40	-30	0	0	0	0	
0	1	2	1	0	0	24	24/1
0	<b>1</b>	0	0	1	0	16	16/1
0	0	1	0	0	1	8	

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## The Simplex Method

A second move:

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant
1	0	-30	0	40	0	640
0	0	2	1	-1	0	8
0	1	0	0	1	0	16
0	0	1	0	0	1	8

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## The Simplex Method

A second move:

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant
1	0	<b>-30</b>	0	40	0	640
0	0	<b>2</b>	1	-1	0	8
0	1	<b>0</b>	0	1	0	16
0	0	<b>1</b>	0	0	1	8

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## The Simplex Method

A second move:

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant	
1	0	-30	0	40	0	640	$30R_2/2 + R_1$
0	0	<b>2</b>	1	-1	0	8	$R_2/2$
0	1	0	0	1	0	16	
0	0	1	0	0	1	8	$R_4 - R_2/2$

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## The Simplex Method

A second move:

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant
1	0	0	15	25	0	760
0	0	1	1/2	-1/2	0	4
0	1	0	0	1	0	16
0	0	0	-1/2	1/2	1	4

New basis:

$$\begin{bmatrix} y & x & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus  $x = 16$ ,  $y = 4$ ,  $s_1 = 0$ ,  $s_2 = 0$ ,  $s_3 = 8$

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## The Simplex Method

A second move:

$p$	$x$	$y$	$s_1$	$s_2$	$s_3$	Constant
1	0	0	15	25	0	760
0	0	1	1/2	-1/2	0	4
0	1	0	0	1	0	16
0	0	0	-1/2	1/2	1	4

New basis:

$$\begin{bmatrix} y & x & s_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus  $x = 16$ ,  $y = 4$ ,  $s_1 = 0$ ,  $s_2 = 0$ ,  $s_3 = 8$

**Since there are no more negative entries in the first row, we have reached the optimal solution**

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## A minimization problem

$$\min x + 4y$$

subject to:

$$x + 2y \geq 8$$

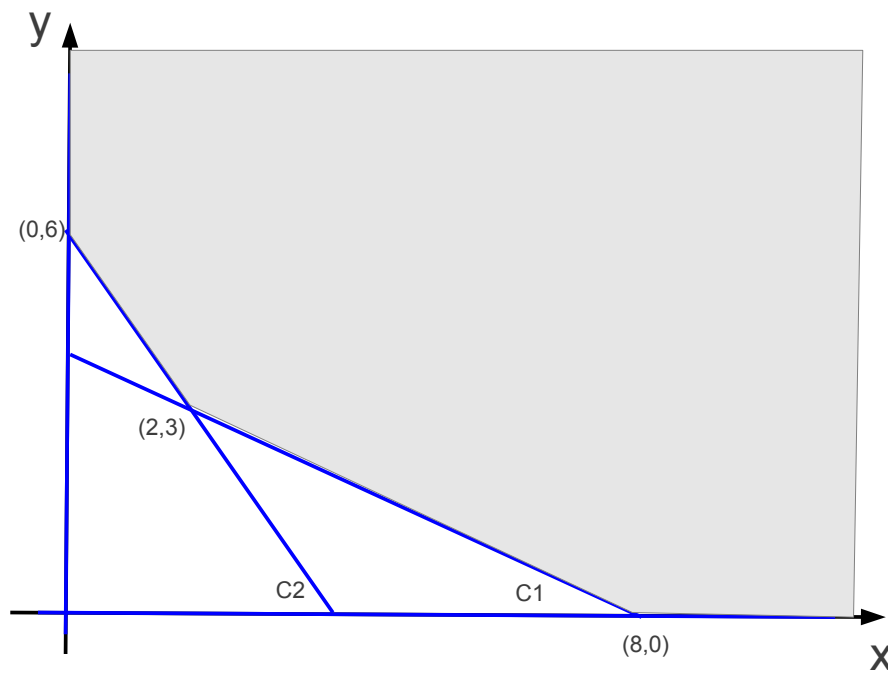
$$3x + 2y \geq 12$$

$$x \geq 0$$

$$y \geq 0$$

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## Graphical view



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## Step 1: Transformation

$$\min x + 4y$$

subject to:

$$x + 2y - s_1 = 8$$

$$3x + 2y - s_2 = 12$$

$$x \geq 0$$

$$y \geq 0$$

$$s_1, s_2 \geq 0$$

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## Step 2: Matrix Form

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

If  $x = y = 0$ , then  $s_1 = -8$ , and  $s_2 = -12$ , which violates the nonnegativity restrictions on  $s_1, s_2$ .

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## Step 1: Transformation

**Add artificial variables  $a_1, a_2$**

$$\min x + 4y + 1000a_1 + 1000a_2$$

subject to:

$$x + 2y - s_1 + a_1 = 8$$

$$3x + 2y - s_2 + a_2 = 12$$

$$x \geq 0$$

$$y \geq 0$$

$$s_1, s_2 \geq 0$$

$$a_1, a_2 \geq 0$$

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## Step 2: Matrix Form

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ 3 & 2 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

If  $x = y = s_1 = s_2 = 0$ , then  $a_1 = 8$ , and  $a_2 = 12$ , which gives us an easy starting point.

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## Step 3: Simplex Tableau

Simplex tableau:

$p$	$x$	$y$	$s_1$	$s_2$	$a_1$	$a_2$	Constant
1	-1	-4	0	0	-1000	-1000	0
0	1	2	-1	0	1	0	8
0	3	2	0	-1	0	1	12

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### Step 3: Simplex Tableau

Simplex tableau:

$p$	$x$	$y$	$s_1$	$s_2$	$a_1$	$a_2$	Constant	
1	-1	-4	0	0	-1000	-1000	0	$1000(R_2 + R_3) + R_1$
0	1	2	-1	0	1	0	8	
0	3	2	0	-1	0	1	12	

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### Step 3: Simplex Tableau

Simplex tableau:

$p$	$x$	$y$	$s_1$	$s_2$	$a_1$	$a_2$	Constant
1	3999	3996	-1000	-1000	0	0	20000
0	1	2	-1	0	1	0	8
0	3	2	0	-1	0	1	12

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## Step 4: Pivoting

For minimization, choose the column associated with largest positive element in  $R_1$ .

$p$	$x$	$y$	$s_1$	$s_2$	$a_1$	$a_2$	Constant
1	<b>3999</b>	3996	-1000	-1000	0	0	20000
0	<b>1</b>	2	-1	0	1	0	8
0	<b>3</b>	2	0	-1	0	1	12

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## Step 4: Pivoting

For minimization, choose the column associated with largest positive element in  $R_1$ .

$p$	$x$	$y$	$s_1$	$s_2$	$a_1$	$a_2$	Constant
1	3999	3996	-1000	-1000	0	0	20000
0	1	2	-1	0	1	0	8
0	<b>3</b>	2	0	-1	0	1	12

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## Step 5: Basis Update

$p$	$x$	$y$	$s_1$	$s_2$	$a_1$	$a_2$	Constant	
1	3999	3996	-1000	-1000	0	0	20000	$R_1 - 3999R_3/3$
0	1	2	-1	0	1	0	8	$R_2 - R_3/3$
0	<b>3</b>	2	0	-1	0	1	12	$R_3/3$

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## Step 5: Basis Update

$p$	$x$	$y$	$s_1$	$s_2$	$a_1$	$a_2$	Constant
1	0	1330	-1000	333	0	-1333	4004
0	0	4/3	-1	1/3	1	-1/3	4
0	1	2/3	0	-1/3	0	1/3	4

New basis:

$$\begin{bmatrix} a_1 & x \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus  $x = 4$ ,  $y = 0$ ,  $s_1 = 0$ ,  $s_2 = 0$ ,  $a_1 = 4$ ,  $a_2 = 0$

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## Step 4: Pivoting

$p$	$x$	$y$	$s_1$	$s_2$	$a_1$	$a_2$	Constant
1	0	<b>1330</b>	-1000	333	0	-1333	4004
0	0	<b>4/3</b>	-1	1/3	1	-1/3	4
0	1	<b>2/3</b>	0	-1/3	0	1/3	4

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## Step 4: Pivoting

$p$	$x$	$y$	$s_1$	$s_2$	$a_1$	$a_2$	Constant
1	0	1330	-1000	333	0	-1333	4004
0	0	<b>4/3</b>	-1	1/3	1	-1/3	4
0	1	2/3	0	-1/3	0	1/3	4

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## Step 5: Basis Update

$p$	$x$	$y$	$s_1$	$s_2$	$a_1$	$a_2$	Constant	
1	0	1330	-1000	333	0	-1333	4004	$R_1 - 1330(3R_2/4)$
0	0	<b>4/3</b>	-1	1/3	1	-1/3	4	$3R_2/4$
0	1	2/3	0	-1/3	0	1/3	4	$R_3 - 2/3(3R_2/4)$

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## Step 5: Basis Update

$p$	$x$	$y$	$s_1$	$s_2$	$a_1$	$a_2$	Constant
1	0	0	-5/2	1/2	-1995/2	-2001/2	14
0	0	1	-3/4	1/4	3/4	-1/4	3
0	1	0	-1/2	-1/2	-1/2	-1/6	2

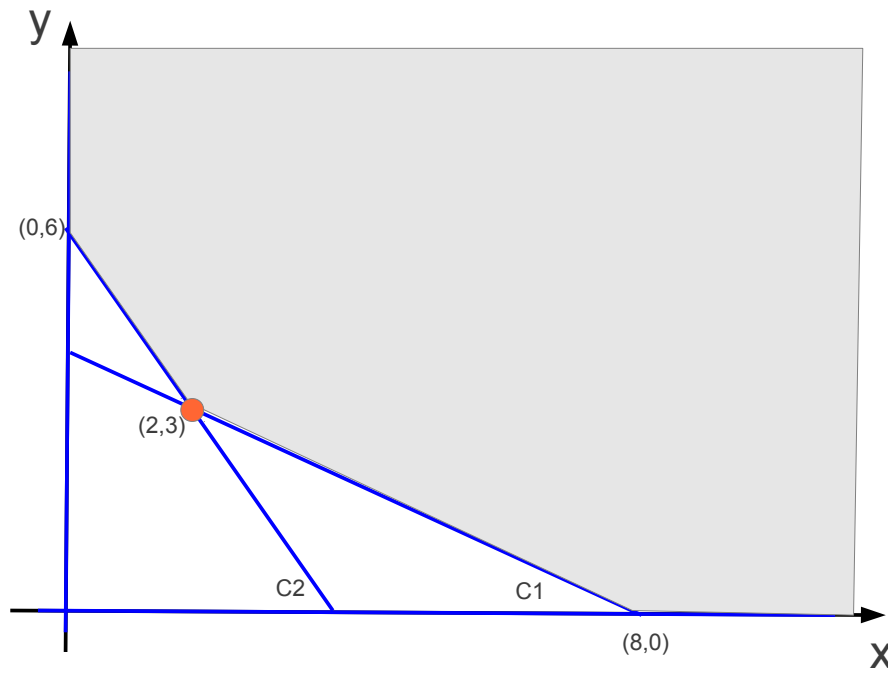
New basis:

$$\begin{bmatrix} y & x \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus  $x = 2$ ,  $y = 3$ ,  $s_1 = 0$ ,  $s_2 = 0$ ,  $a_1 = 0$ ,  $a_2 = 0$

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## Graphical view



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Finish the example

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