# MATH529 – Fundamentals of Optimization
# Duality, Game Theory and Linear Programming

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## Economic Interpretation of a Dual

Back to the diet problem:

Food 1: \$0.6 cts per 100 g.
Food 2: \$1 cts per 100 g.

| Nutrient | Food 1 | Food 2 | Minimum Daily Requirement |
|---|---|---|---|
| Calcium | 10 | 4 | 20 |
| Protein | 5 | 5 | 20 |
| Vitamins | 2 | 6 | 12 |

## Economic Interpretation of a Dual

Primal problem:

$$\text{Minimize } C = 0.6x + y$$

subject to:

$$10x + 4y \geq 20$$
$$5x + 5y \geq 20$$
$$2x + 6y \geq 12$$
$$x, y \geq 0$$



## Economic Interpretation of a Dual

Dual problem:

$$\text{Maximize } V = 20u + 20v + 12w$$

subject to:

$$10u + 5v + 2w \leq 0.6$$
$$4u + 5v + 6w \leq 1$$
$$u, v, w \geq 0$$

## Economic Interpretation of a Dual

Dimensional analysis of the primal problem:

- $x, y$ are in units of 100g, that is, hg.
- Coefficients of objective function are in \$/hg
- Coefficient matrix is in (nutritional content/hg).

Dimensional analysis of the dual problem:

- $u, v, w$ are expressed in (\$/nutritional content) units.
- Coefficients of objective function are in nutritional content units.
- Coefficient matrix is in (nutritional content/hg).



## Duality and Game Theory

# Intro

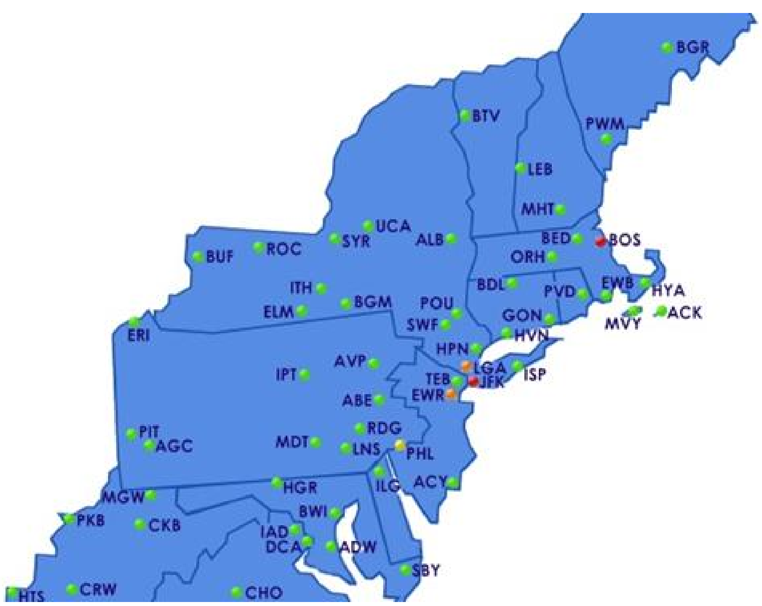
# Duality and Game Theory

Game Theory is a branch of mathematics that analyzes competitive situations in which the outcome depends on the decisions of all the participants.

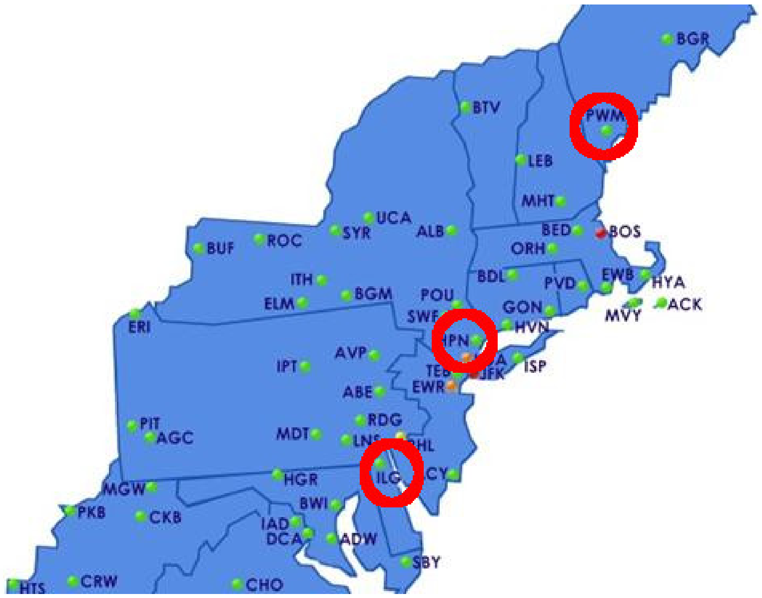
2-person zero-sum games are situations in which one player gains at the expence of the other.



# Duality and Game Theory



## Duality and Game Theory




ILG - New Castle, DE
HPN - Westchester, NY
PWN - Portland, ME



## Duality and Game Theory

| Airport | Probability of capture |
|---|---|
| ILG | 1/2 |
| HPN | 3/4 |
| PWN | 1/3 |

- **Pure strategy:** Example: DL's choice $\longrightarrow$ PWN (always), DEA $\longrightarrow$ PWN (always). Expected drug in the country: 2/3 of total shipment.

## Duality and Game Theory

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- **Pure strategy:** Example: DL's choice $\longrightarrow$ PWN (always), DEA $\longrightarrow$ PWN (always). Expected drug in the country: 2/3 of total shipment.
- **Mixed strategy:** Example: DL's choice $\longrightarrow$ ILG, HPN, PWN at random ($p = 1/3$ for each airport), DEA $\longrightarrow$ HPN. Expected drug in the country: 3/4 of total shipment.



## Duality and Game Theory

Players: DL (Row), DEA (Column).

Payoff matrix (from the point of view of the DL):

| | DEA | | | |
|---|---|---|---|---|
| | | ILG | HPN | PWM |
| | ILG | 1/2 | 1 | 1 |
| DL | HPN | 1 | 1/4 | 1 |
| | PWN | 1 | 1 | 2/3 |

Expected payoffs if DL goes to all airports with equal probability:

- If DEA goes to ILG:
- If DEA goes to HPN:
- If DEA goes to PWN:

## Duality and Game Theory

Are there any optimal strategies for both players?

Can the DL have a guaranteed payoff regardless of what the DEA does?



## Duality and Game Theory

Worst case scenario for DL: DL goes to each airport with equal probability (that is, he does not use any information about probability of being intercepted), the DEA knows this and therefore the DEA maximizes its chances of intercepting the DL's shipment.

Then, the lower bound on the DL's payoff is:

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Worst case scenario for DEA: DEA goes to each airport with equal probability (that is, it does not use any information about probability of intercepting a shipment), the DL knows this and therefore the DL minimizes the chances of being intercepted.

Then, the upper bound on the DL's payoff is:



## Duality and Game Theory

A *mixed strategy* of player C is a probability distribution $\mathbf{x}$ over the set of columns $\{1, \ldots, n\}$. Similarly, a mixed strategy of player R is a probability distribution $\mathbf{y}$ over the set of rows $\{1, \ldots, m\}$.

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The set of all possible mixed strategies of player C is $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{j=1}^{n} x_j = 1,\ x_j \geq 0\}$ and the set of possible mixed strategies of player R is $\mathcal{Y} = \{\mathbf{y} \in \mathbb{R}^m \mid \sum_{i=1}^{m} y_i = 1,\ y_i \geq 0\}$.



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If the R player chooses row $i$ of the payoff matrix, then the expected payoff is

$$\sum_{j=1}^{n} p_{ij} x_j = \mathbf{p}_i^T \mathbf{x}$$

E.g. DL $\longrightarrow$ ILG $(i = 1)$, $\mathbf{p}_i^T \mathbf{x} = (1/2)(x_1) + 1(x_2) + 1(x_3)$

## Duality and Game Theory

Thus, if the players use mixed strategies $\mathbf{x}$ (C) and $\mathbf{y}$ (R), respectively, the probability of payoff $p_{ij}$ is $y_i x_j$ and the expected payoff is $\mathbf{y}^T P \mathbf{x}$.



## Duality and Game Theory

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Minimize the worst possible outcome, that is:

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \mathbf{y}^T P\mathbf{x} = \min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} (P\mathbf{x})^T\mathbf{y}.$$



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In our example, if the DEA's mixed strategy is $\mathbf{x} = (1/3, 1/3, 1/3)^T$, then $P\mathbf{x} = (5/6, 3/4, 8/9)^T$ and the DL will choose $\mathbf{y} = (y_1, y_2, y_3)^T$ that solves:

$$\text{Maximize } (5/6)y_1 + (3/4)y_2 + (8/9)y_3$$

subject to:

$$y_1 + y_2 + y_3 = 1$$

$$y_i \geq 0$$

## Duality and Game Theory

Note that the solution to the program above has the form $\mathbf{y}^\star = \mathbf{e}_i$, where $\mathbf{e}_i$ is the vector with a 1 in the $i$-th component and zero everywhere else.

Therefore, we can say that

$$\max_{\mathbf{y}\in\mathcal{Y}} (P\mathbf{x})^T\mathbf{y} = \max_{1\le i\le m} (P\mathbf{x})_i$$



## Duality and Game Theory

Now, the problem the DEA has to solve is:

$$\text{Minimize } v$$

$$\text{subject to:}$$

$$\begin{aligned} (1/2)x_1 + x_2 + x_3 &\le v \\ x_1 + (1/4)x_2 + x_3 &\le v \\ x_1 + x_2 + (2/3)x_3 &\le v \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

$$x_1, x_2, x_3 \ge 0$$

## Duality and Game Theory

More compactly, the DEA has to solve the LP:

Minimize $v$

subject to:

$$P\mathbf{x} \leq v\mathbf{e}$$
$$\mathbf{x} \in \mathcal{X}$$

where $\mathbf{e}$ is a vector with all its components equal to one and length $n$.

Using MATLAB, the solution is:

```
x = linprog(f,A,b,Aeq,beq,lb,ub)
```



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where $\mathbf{e}$ is a vector with all its components equal to one and length $n$.

Using MATLAB, the solution is: $x_1 = 0.3158$, $x_2 = 0.2105$, $x_3 = 0.4737$, and $v = 0.8421$.

## Duality and Game Theory

How can the DL maximize the amount of drug that enters into the country?



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$$\max_{\mathbf{y}\in\mathcal{Y}} \min_{\mathbf{x}\in\mathcal{X}} \mathbf{y}^T P\mathbf{x} = \max_{\mathbf{y}\in\mathcal{Y}} \min_{\mathbf{x}\in\mathcal{X}} (P^T\mathbf{y})^T\mathbf{x}.$$

## Duality and Game Theory

How can the DL maximize the amount of drug that enters into the country?

Maximize the worst possible outcome, that is:

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In our example, if the DEA's mixed strategy is $\mathbf{y} = (1/3, 1/3, 1/3)^T$, then $P^T\mathbf{y} = (5/6, 3/4, 8/9)^T$ and the DEA will choose $\mathbf{x} = (x_1, x_2, x_3)^T$ that solves:

$$\text{Minimize } (5/6)x_1 + (3/4)x_2 + (8/9)x_3$$

$$\text{subject to:}$$

$$x_1 + x_2 + x_3 = 1$$

$$x_j \geq 0$$



## Duality and Game Theory

Again, note that the solution to the program above has the form $\mathbf{x}^\star = \mathbf{e}_j$, where $\mathbf{e}_j$ is the vector with a 1 in the $j$-th component and zero everywhere else.

Thus, we can say that

$$\min_{\mathbf{x}\in\mathcal{X}} (P^T\mathbf{y})^T\mathbf{x} = \min_{1\leq j\leq n} (P^T\mathbf{y})_j$$

## Duality and Game Theory

The DL solves then the following LP:

$$\text{Maximize } u$$

$$\text{subject to:}$$

$$\begin{aligned}(1/2)y_1 + y_2 + y_3 &\geq u \\ y_1 + (1/4)y_2 + y_3 &\geq u \\ y_1 + y_2 + (2/3)y_3 &\geq u \\ y_1 + y_2 + y_3 &= 1\end{aligned}$$

$$y_1, y_2, y_3 \geq 0$$



## Duality and Game Theory

More compactly, the DL has to solve the LP:

$$\text{Maximize } u$$

$$\text{subject to:}$$

$$\begin{aligned}P^T\mathbf{y} &\geq u\mathbf{e} \\ \mathbf{y} &\in \mathcal{Y}\end{aligned}$$

where $\mathbf{e}$ is a vector with all its components equal to one and length $m$.

Since this problem is the dual of the problem solved by the DEA, the solution is: $y_1 = 0.3158$, $y_2 = 0.2105$, $y_3 = 0.4737$, and $u = 0.8421$.

# Duality and Game Theory

The theory of duality and the properties of linear programs lead us to conclude that the optimal values of the DEA and DL problems ($\mathbf{x}^\star$ and $\mathbf{y}^\star$, respectively) are identical and form a saddle point of the Lagrangian:

$$\mathbf{y}^T P \mathbf{x}^\star \leq (\mathbf{y}^\star)^T P \mathbf{x}^\star \leq (\mathbf{y}^\star)^T P \mathbf{x}$$

The point $(\mathbf{x}^\star, \mathbf{y}^\star)$ is an equilibrium, which means that it is not profitable for any player to deviate from his strategy.