# MATH529 – Fundamentals of Optimization
# Unconstrained Optimization I

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# Before we start: Syllabus Info!

# Meetings & Contact

| | |
|---|---|
| Instructor: | Dr. Marco A. Montes de Oca |
| Office: | 315 Ewing Hall |
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| URL: | `http://math.udel.edu/~mmontes/teaching/UD/S14-MATH529-10.html`<br>`https://sakai.udel.edu/portal/` |
| Office hours: | Mondays 5:00pm–7:00pm or by appointment |
| Meetings: | Mondays and Wednesdays 3:35pm–4:50pm, 330 Purnell Hall |

# Evaluation

The final grade components are: Homeworks, Exams, and a Project.

The contribution of each component is as follows:

| Component | Weight |
| --- | --- |
| Homeworks | 30% |
| Exam 1 | 15 % |
| Exam 2 | 15 % |
| Final Exam | 20 % |
| Project | 20 % |

Back to business.

# Basics

# Problem Statement

Given a set $X$ and a function $f : X \to \mathbb{R}$, find an element $x^\star \in X$ such that

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Typically, $X \subseteq \mathbb{R}^n$ and $f$ will be relatively nice (e.g., differentiable).

The definition of $X$ will be based on systems of equations and inequalities called *constraints*.

# Notation

The standard notation used to represent an optimization problem is:

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$$

subject to

$$c_i(\mathbf{x}) = 0, \quad i \in \mathcal{E}$$

$$c_i(\mathbf{x}) \geq 0 \text{ (or } c_i(\mathbf{x}) \leq 0), \quad i \in \mathcal{I}$$

where the functions $c_i(\mathbf{x}),\ i \in \mathcal{E}$ are the *equality constraints*, and the functions $c_i(\mathbf{x}),\ i \in \mathcal{I}$ are the *inequality constraints*.

# Example

Using the "studying for finals" problem:

$$\max_{\mathbf{x}\in\mathbb{R}^5} \; 20\left(\left(\frac{x_1}{x_1+1}\right)^2 + \left(\frac{x_2}{x_2+1}\right)^{1.7} + \left(\frac{x_3}{x_3+1}\right)^{1.8} + \left(\frac{x_4}{x_4+1}\right)^{2.5} + \left(\frac{x_5}{x_5+1}\right)^{0.5}\right)$$

subject to

$$c_1(\mathbf{x}) = \sum_{i=1}^{5} x_i \le 22\,,$$

$$0 \le x_i \le 22\,,\; i \in \{1,2,3,4,5\}\,.$$

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# Unconstrained Optimization Problems

# Functions of one variable

Solving the unconstrained optimization problem

$$\min_{x \in \mathbb{R}} \; f(x)$$

means finding a point $x^\star \in \mathbb{R}$ such that $f(x^\star) \leq f(x)$ for all $x \in \mathbb{R}$.

Two questions always arise:

- Given a point $x^c$, how can we know whether $f(x^c) \leq f(x)$ for all $x \in \mathbb{R}$, and therefore that $x^c = x^\star$?
- How can we find $x^\star$ if we know only $f(x)$ and possibly its derivatives?

The fundamental tool we are going to use is *Taylor's formula*:

**Theorem (Taylor's formula or the Extended Law of the Mean)**

*Suppose that* $f(x)$, $f'(x)$, $f''(x)$ *exist on the closed interval* $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. *If* $x^\star$, $x$ *are any two different points of* $[a, b]$, *then there exists a point* $z$ *strictly between* $x^\star$ *and* $x$ *such that*

$$f(x) = f(x^\star) + f'(x^\star)(x - x^\star) + \frac{f''(z)}{2}(x - x^\star)^2 . \qquad (1)$$

Derivation of Taylor's formula

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If $f''(x) > 0$ for all $x \in \mathbb{R}$ and $f'(x^\star) = 0$, then from Eq. 1:

$$f(x) = f(x^\star) + 0 + \text{positive number}\,, \text{for all } x \neq x^\star$$

so

$$f(x) - f(x^\star) > 0 \Rightarrow f(x) > f(x^\star) \text{ for all } x \neq x^\star$$

Thus, $x^\star$ minimizes $f(x)$.

# Optimization via Calculus

## Example

Show that $x = 0$ minimizes the value of $f(x) = e^{x^2}$.

# Optimization via Calculus

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Suppose $f(x)$ is a real-valued function defined on some interval $I$. A point $x^\star$ in $I$ is:

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- A **local minimizer** for $f(x)$ if there is a $\delta > 0$ such that $f(x^\star) \leq f(x)$ for all $x$ in $I$ for which $x^\star - \delta < x < x^\star + \delta$;

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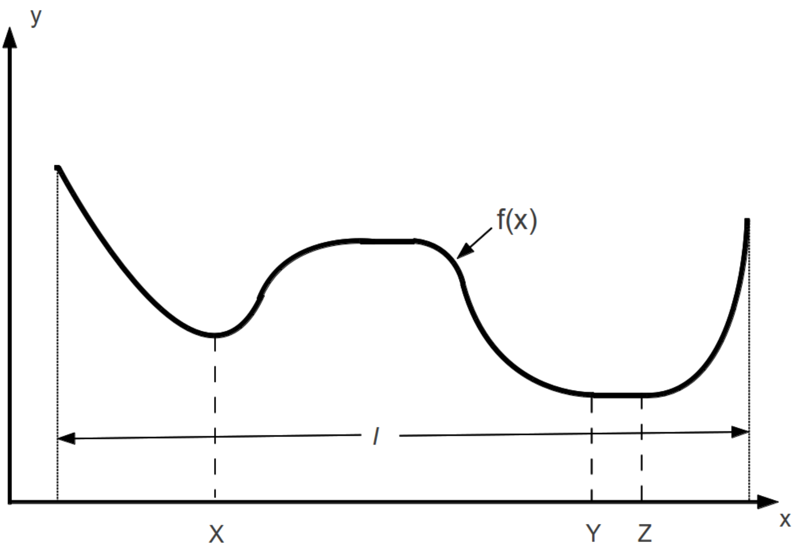
# Optimization via Calculus



# Optimization via Calculus

Two theorems summarize the basic facts about optimization of one variable functions.

**Theorem (Local minimizer identification)**

*Suppose that $f(x)$ is a differentiable function on an interval $I$. If $x^\star$ is a local minimizer of $f(x)$, then $f'(x^\star) = 0$.*

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**Theorem (Local minimizer identification)**

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**Theorem (Classification of minimizers)**

*Suppose that $f(x)$, $f'(x)$, and $f''(x)$ are all continuous on an interval $I$ and that $x^\star \in I$ is a critical point of $f(x)$.*

*a) If $f''(x) \geq 0$ for all $x \in I$, then $x^\star$ is a global minimizer of $f(x)$ on $I$.*
*b) If $f''(x) > 0$ for all $x \in I$ such that $x \neq x^\star$, then $x^\star$ is a strict global minimizer of $f(x)$ on $I$.*
*c) If $f''(x^\star) > 0$, then $x^\star$ is a strict local minimizer of $f(x)$.*

**Example**

Find the local and global minimizers and maximizers on $\mathbb{R}$ of $f(x) = 3x^4 - 4x^3 + 1$.

To be continued . . . .