MATH529 – Fundamentals of Optimization
Unconstrained Optimization IV

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Recap

Algorithmic Approaches

Why do we need them?

Because:

- Obtaining a closed form for $\mathbf{x}^\star$ such that $\nabla f(\mathbf{x}^\star) = \mathbf{0}$ may be extremely hard, or even impossible.
- Analyzing the Hessian of f to determine the nature of stationary points is usually harder than solving $\nabla f(\mathbf{x}^\star) = \mathbf{0}$.

In conclusion, Calculus gives us extremely powerful theoretical insights, but not very practical tools.

Algorithmic Approaches

The main goal :

Generate a sequence of points $\{\mathbf{x}_k\}$ such that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}^\star$ (As fast as possible).

The merits of an optimization algorithm depend on:

- Cost: How much computational effort is needed to generate the points in the sequence $\{\mathbf{x}_k\}$?
- Convergence Rate: How fast does the sequence $\{\mathbf{x}_k\}$ converge?
- Convergence Guarantees: Does the sequence $\{\mathbf{x}_k\}$ converge to a fixed point only, to a stationary point, or to a minimizer?

Algorithmic Approaches

Two fundamental methods:

- Newton's Method
- Steepest Descent Method

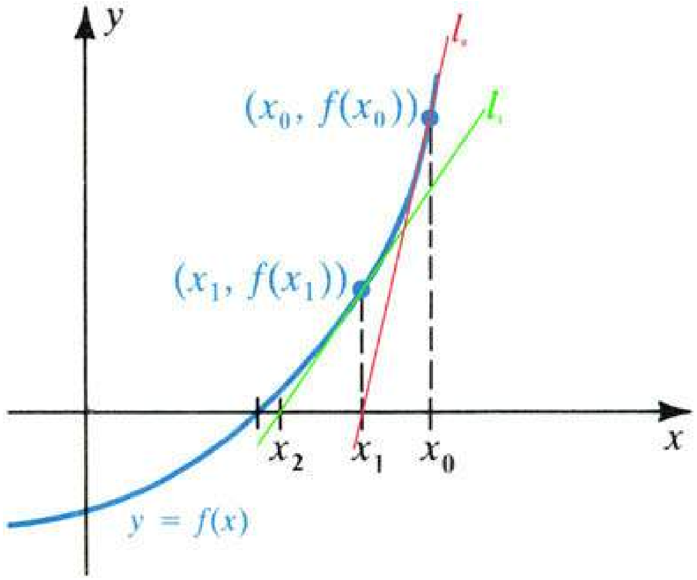
Newton's Method

Two interpretations:

- Root finder
- Optimization via a surrogate model

Newton's Method as a Root Finder

In 1D:



Newton's Method as a Root Finder

In 1D:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

So:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Newton's Method as a Root Finder

In n dimensions:

$\nabla f(\mathbf{x}^\star) = \mathbf{0}$, where $\mathbf{x}^\star \in \mathbb{R}^n$, is actually a system of equations of the form

$$\nabla f(\mathbf{x}^\star) = \mathbf{g}(\mathbf{x}^\star) \begin{pmatrix} g_1(\mathbf{x}^\star) \\ g_2(\mathbf{x}^\star) \\ g_3(\mathbf{x}^\star) \\ \vdots \\ g_n(\mathbf{x}^\star) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{1}$$

Newton's Method as a Root Finder

Using Newton's method to approximate $\mathbf{x}^\star$:

$$\begin{pmatrix} g_1(\mathbf{x}_k) + \nabla g_1(\mathbf{x}_k)^T(\mathbf{x} - \mathbf{x}_k) \\ g_2(\mathbf{x}_k) + \nabla g_2(\mathbf{x}_k)^T(\mathbf{x} - \mathbf{x}_k) \\ g_3(\mathbf{x}_k) + \nabla g_3(\mathbf{x}_k)^T(\mathbf{x} - \mathbf{x}_k) \\ \vdots \\ g_n(\mathbf{x}_k) + \nabla g_n(\mathbf{x}_k)^T(\mathbf{x} - \mathbf{x}_k) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2}$$

Or in matrix notation:

$$\mathbf{g}(\mathbf{x}_k) + \boldsymbol{\nabla}\mathbf{g}(\mathbf{x}_k)(\mathbf{x} - \mathbf{x}_k) \tag{3}$$

where $\boldsymbol{\nabla}\mathbf{g}(\mathbf{x}_k)$ is the Jacobian matrix of $\mathbf{g}(\mathbf{x})$.

Newton's Method as a Root Finder

Therefore:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - (\nabla \mathbf{g}(\mathbf{x}_k))^{-1} \mathbf{g}(\mathbf{x}_k) \tag{4}$$

Newton's Method as a Root Finder

Example

"Run" two iterations of Newton's method to find a root of:

$$x^2 + y^2 + z^2 = 3$$
$$x^2 + y^2 - z = 1$$
$$x + y + z = 3$$

starting at $\mathbf{x}_0 = (1, 0, 1)^T$