MATH529 – Fundamentals of Optimization
Unconstrained Optimization VII

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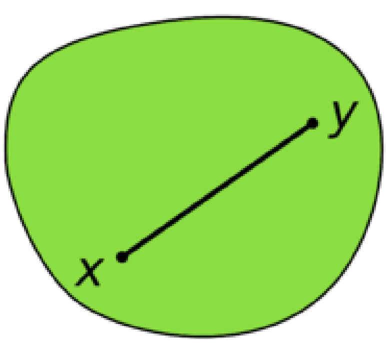
Recap

Part I: Convex sets and functions

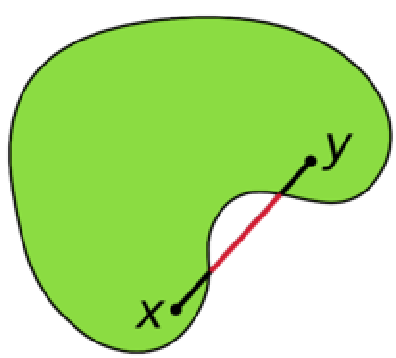
Convex Sets

Definition (Convex Set)

A set C in $\mathbb{R}^n$ is convex if for every $\mathbf{x}$ and $\mathbf{y}$ in C, the line segment $t\mathbf{x} + (1-t)\mathbf{y}$, with $t \in [0,1]$, also lies in C.



Convex



Non-convex

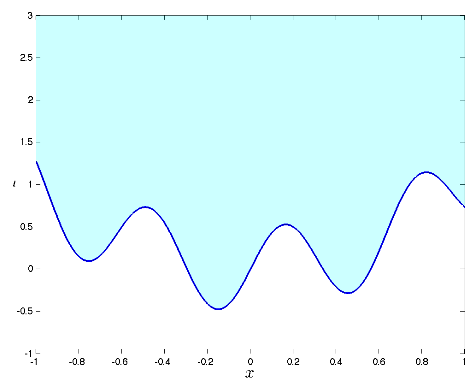
By convention, a single point $\mathbf{x}$ and the null set are also convex.

Convex Functions

Definition (Epigraph)

Associated with every function $f : \mathbb{R}^n \to \mathbb{R}$ there is a set called the epigraph of f, denoted by $epi(f)$, defined as

$$epi(f) = \{(\mathbf{x}, v) \in \mathbb{R}^n \times \mathbb{R} \text{ such that } v \geq f(\mathbf{x})\}$$

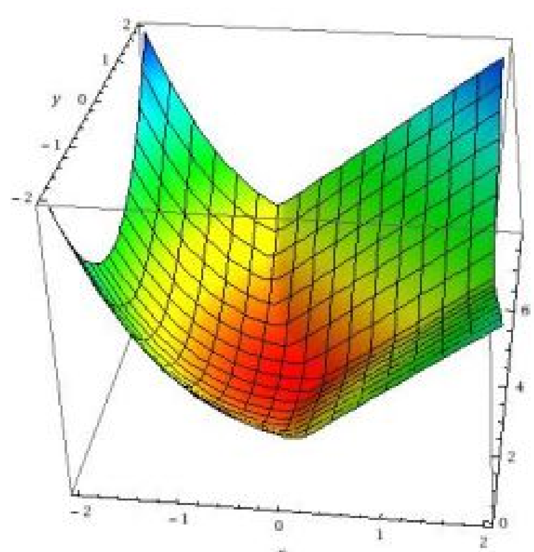


Taken from `https://inst.eecs.berkeley.edu/~ee127a/book/login/Images/intro_graphs.png`

Convex Functions

Definition (Convex Function)

A function f is called convex if $epi(f)$ is a convex set.



Taken from `http://cssanalytics.files.wordpress.com/2013/08/convex-function.png?w=670`

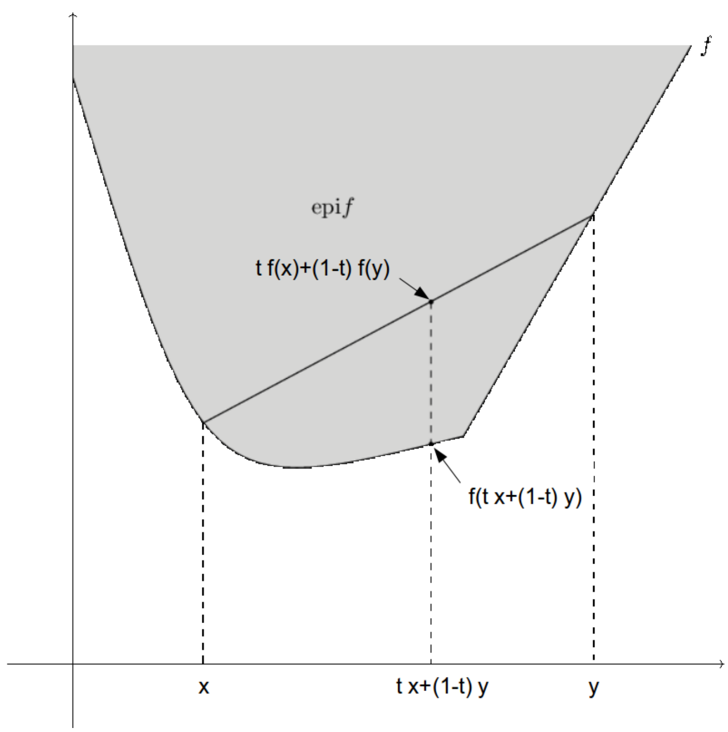
Convex Functions

Let $(\mathbf{x}_1, f(\mathbf{x}_1)) \in epi(f)$ and $(\mathbf{x}_2, f(\mathbf{x}_2)) \in epi(f)$. If f is convex, then

$$t(\mathbf{x}_1, f(\mathbf{x}_1) + (1-t)(\mathbf{x}_2, f(\mathbf{x}_2) = (t\mathbf{x}_1 + (1-t)\mathbf{x}_2, tf(\mathbf{x})_1 + (1-t)f(\mathbf{x}_2))$$

also belongs to $epi(f)$ for $t \in [0, 1]$.

Convex Functions



Convex Functions

Definition (Convex Function)

An alternative definition of a convex function is that if for all $\mathbf{x}_1$ and $\mathbf{x}_2$ and all $0 \leq t \leq 1$ we have that

$$f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) \leq tf(\mathbf{x}_1) + (1-t)f(\mathbf{x}_2)$$

If $f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) < tf(\mathbf{x}_1) + (1-t)f(\mathbf{x}_2)$, $\mathbf{x}_1 \neq \mathbf{x}_2$, then f is called strictly convex.

Convex Functions

Example

Show that function $f(x) = x^2$ is convex.

Convexity and Differentiability

Theorem

Suppose that f has continuous first partial derivatives on a convex set D. The function f is convex if and only if

$$f(\mathbf{x}_2) \geq f(\mathbf{x}_1) + \nabla f(\mathbf{x}_1)^T(\mathbf{x}_2 - \mathbf{x}_1)$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in D$.

Convexity and Differentiability

Corollary

If f is a convex function with continuous 1st partial derivatives on some convex set D, then any critical point of f on D is a global minimizer of f.

If f is convex, then

$$f(\mathbf{x}) \geq f(\mathbf{x}^\star) + \nabla f(\mathbf{x}^\star)(\mathbf{x} - \mathbf{x}^\star)$$

where $\mathbf{x}, \mathbf{x}^\star \in D$.
If $\mathbf{x}^\star$ is a critical point of f, then $\nabla f(\mathbf{x}^\star) = 0$ and therefore

$$f(\mathbf{x}) \geq f(\mathbf{x}^\star)$$

for all $\mathbf{x} \in D$.

Convexity and Differentiability

Theorem

Suppose that f has continuous second partial derivatives on D. If the Hessian of f, Hf, is positive semidefinite on D, then f is convex on D.