University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C

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Spring 2012

Exam I

Name: ______________________________ **Section:** 51

March 5, 2012

Question	1	2	3	4	5	Bonus	Total
Points							

Instructions

- The exam is composed of **five** problems for a total of 100 points, plus a bonus problem for 10 extra points.
- Read very carefully each problem before working on it.
- Partial credit will not be given if appropriate work is not shown.
- If you get stuck on a problem, skip it and come back to it if you have extra time at the end.
- Answer questions in the space provided. If you need more space for an answer, continue your answer on the back of the page, or/and use the margins of the test pages.
- Carefully work out each problem and clearly indicate your final answer to any problem.
- You may **not** use calculators, dictionaries, notes, or any other kinds of aids.
- **DISHONESTY WILL NOT BE TOLERATED.**

Problems

1. [20 points] The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are given in terms of the standard basis vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ by the relations

$$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}, \qquad \vec{b} = \hat{i} - 2\hat{j} + 2\hat{k}, \qquad \vec{c} = -2\hat{i} + \hat{j} - 2\hat{k}$$

Suppose $\vec{F} = 3\hat{i} - \hat{j} + 2\hat{k}$. Express $\vec{F}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$; that is, find the scalars α, β, γ, such that $\vec{F} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$.

Solution: If $\vec{F} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$, then $\vec{F} = \alpha(2\hat{i} + 3\hat{j} - \hat{k}) + \beta(\hat{i} - 2\hat{j} + 2\hat{k}) + \gamma(-2\hat{i} + \hat{j} - 2\hat{k})$. Thus, $\vec{F} = 3\hat{i} - \hat{j} + 2\hat{k} = (2\alpha + \beta - 2\gamma)\hat{i} + (3\alpha - 2\beta + \gamma)\hat{j} + (-\alpha + 2\beta - 2\gamma)\hat{k}$. This means that

$2\alpha + \beta - 2\gamma = 3,$
$3\alpha - 2\beta + \gamma = -1,$
$-\alpha + 2\beta - 2\gamma = 2.$

Solving this system of equations, we find that $\alpha = 2$, $\beta = 5$, and $\gamma = 3$. Thus, $\vec{F} = 2\vec{a} + 5\vec{b} + 3\vec{c}$.

2. [20 points] Find an equation of the line that passes through the origin and the point $(2, 2, 1)$. What are the angles that the line makes with the coordinate axes?

Solution: If we take the origin as a reference point and the vector connecting the origin with $(2, 2, 1)$ as direction vector, then an equation of the line that passes through $(0, 0, 0)$ and $(2, 2, 1)$ is

$\vec{r}(t) =< 0, 0, 0 > +t < 2 - 0, 2 - 0, 1 >=< 2t, 2t, t >$. Or parametrically, $x = 2t$,$y = 2t$, and $z = t$.

The angles that the line makes with the coordinate axes are the same that the direction vector $\vec{v} =< 2, 2, 1 >$ makes with the axes. The angle between $\vec{v}$ and the x-axis, is equal to the angle that $\vec{v}$ makes with the standard basis vector $\hat{i}$. This angle is $\alpha = \cos^{-1}(\frac{\vec{v}\cdot\hat{i}}{|\vec{v}||\hat{i}|}) = \cos^{-1}(\frac{2}{\sqrt{2^2+2^2+1^2}}) = \cos^{-1}(\frac{2}{3})$. The same idea is used for the other the y- and z-axes. The angles the line makes with the axes are then $\cos^{-1}(\frac{2}{3})$, $\cos^{-1}(\frac{2}{3})$, and $\cos^{-1}(\frac{1}{3})$ with the x-, y- and z-axis, respectively.

3. [20 points] Find an equation of the plane that passes through the points $A(1, 0, 0)$, $B(2, 0, -1)$, and $C(1, 4, 3)$. Find the area of the triangle ABC.

Solution: Let use the vectors $\vec{AB} =< 2 - 1, 0 - 0, -1 - 0 >=< 1, 0, -1 >$ and $\vec{AC} =< 1 - 1, 4 - 0, 3 - 0 >=< 0, 4, 3 >$ to find the normal vector of the plane. This normal vector is

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 4 & 3 \end{vmatrix} = 4\hat{i} - 3\hat{j} + 4\hat{k}$$

If we take the vector $\vec{A} =< 1, 0, 0 >$ as reference point, then an equation of the plane is $(\vec{r} - \vec{A}) \cdot \vec{n} = 0$, where $\vec{r} =< x, y, z >$ or, equivalently, $4x - 3y + 4z - 4 = 0$.

The area of the triangle ABC is equal to half the area of the parallelogram spanned by the vectors $\vec{AB}$ and $\vec{AC}$, which is equal to the norm of $|\vec{AB} \times \vec{AC}|$. Thus, the area of the triangle is $\frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{4^2 + (-3)^2 + 4^2} = \frac{1}{2}\sqrt{41}$.

4. [20 points] Show that the planes $x + y - z = 1$ and $2x - 3y + 4z = 5$ are neither parallel nor perpendicular.

   Solution: The angle between two planes is equal to the angle between their normal vectors. If two planes are parallel, their normal vectors should be parallel. If two planes are perpendicular, their normal vectors should be perpendicular. Thus, to test whether two planes are parallel or perpendicular, we use the cross and dot products of their normal vectors.

   The normal vector of the first plane is $\vec{n}_1 =< 1, 1, -1 >$. The normal vector of the second plane is $\vec{n}_2 =< 2, -3, 4 >$. If the planes are parallel, then $|\vec{n}_1 \times \vec{n}_2| = 0$. The cross product $\vec{n}_1 \times \vec{n}_2 = \hat{\imath} - 6\hat{\jmath} - 5\hat{k} =< 1, -6, -5 >$. $|\vec{n}_1 \times \vec{n}_2| = \sqrt{1^1 + (-6)^2 + (-5)^2} = \sqrt{1 + 36 + 25} = \sqrt{62}$. Since $|\vec{n}_1 \times \vec{n}_2| \neq 0$, the planes are not parallel.

   If $\vec{n}_1 \cdot \vec{n}_2 = 0$, then the vectors, and the planes, are perpendicular. The dot product $\vec{n}_1 \cdot \vec{n}_2 =< 1, 1, -1 > \cdot < 2, -3, 4 >= 2 - 3 - 4 = -5$. Again, since $\vec{n}_1 \cdot \vec{n}_2 \neq 0$, then the planes are not perpendicular.

5. [20 points] A surface is described by the vector equation $(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 1$, where $\vec{r} =< x, y, z >$, $\vec{a} =< 1, 1, 1 >$, and $\vec{b} =< 3, 3, 3 >$. Find an algebraic equation for this surface and identify it.

   Solution:

   The equation $(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 1$ is equivalent to $(< x, y, z > - < 1, 1, 1 >) \cdot (< x, y, z > - < 3, 3, 3 >) = 1$. Expanding it, we get $(x-1)(x-3) + (y-1)(y-3) + (z-1)(z-3) = (x^2 - 4x + 3) + (y^2 - 4y + 3) + (z^2 - 4z + 3) = (x^2 - 4x) + (y^2 - 4y) + (z^2 - 4z) = 1 - 3 - 3 - 3 = -8$. Completing squares: $(x^2 - 4x + 4) + (y^2 - 4y + 4) + (z^2 - 4z + 4) = (x-2)^2 + (y-2)^2 + (z-2)^2 = -8 + 4 + 4 + 4 = 4$. Thus, the surface is a sphere centered at the point $(2, 2, 2)$ with radius 2.

6. [Bonus problem: 10 points] Show that the scalar projection of a vector $\vec{A} = \langle a_1, a_2, a_3 \rangle$ onto another vector $\vec{B} = \langle b_1, b_2, b_3 \rangle$ $(\text{comp}_{\vec{B}} \vec{A})$ is equal to $\hat{b} \cdot \vec{A}$, where $\hat{b}$ is a unit vector with the same direction as $\vec{B}$.

   Solution:
   $\text{comp}_{\vec{B}} \vec{A} = \frac{\vec{B} \cdot \vec{A}}{|\vec{B}|} = \frac{1}{|\vec{B}|} (\vec{B} \cdot \vec{A}) = \left(\frac{\vec{B}}{|\vec{B}|} \right) \cdot \vec{A} = \hat{b} \cdot \vec{A}$.