University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C
Instructor: Marco A. Montes de Oca
Spring 2012

Exam II

Name: ______________________________ **Section:** ________

April 9 & 10, 2012

Question	1	2	3	4	5	Bonus	Total
Points							

Instructions

- The exam is composed of **five** problems for a total of 100 points, plus a bonus problem for 10 extra points.
- Read very carefully each problem before working on it.
- Partial credit will not be given if appropriate work is not shown.
- If you get stuck on a problem, skip it and come back to it if you have extra time at the end.
- Answer questions in the space provided. If you need more space for an answer, continue your answer on the back of the page, or/and use the margins of the test pages.
- Carefully work out each problem and clearly indicate your final answer to any problem.
- You may **not** use calculators, dictionaries, notes, or any other kinds of aids.
- **DISHONESTY WILL NOT BE TOLERATED.**

Problems

1. [20 points] Suppose $\vec{A}(t) = \langle 3t^2, -t-4, t^2-2t\rangle$ and $\vec{B}(t) = \langle \sin(t), 3e^{-t}, -3\cos(t)\rangle$. Find the value of the second order derivative of $\vec{A}(t) \times \vec{B}(t)$ with respect to t when $t = 0$.

$\frac{d^2}{dt^2}\left(\vec{A}\times\vec{B}\right) = \frac{d}{dt}\left(\frac{d}{dt}\left(\vec{A}\times\vec{B}\right)\right) = \frac{d}{dt}\left(\vec{A}'\times\vec{B}+\vec{A}\times\vec{B}'\right) = \frac{d}{dt}\left(\vec{A}'\times\vec{B}\right)+\frac{d}{dt}\left(\vec{A}\times\vec{B}'\right) = \vec{A}''\times\vec{B}+2\vec{A}'\times\vec{B}'+\vec{A}\times\vec{B}''$

$\vec{A}|_{t=0} = \langle 0,-4,0\rangle$
$\vec{A}' = \langle 6t,-1,2t-2\rangle|_{t=0} = \langle 0,-1,-2\rangle$
$\vec{A}'' = \langle 6,0,2\rangle|_{t=0} = \langle 6,0,2\rangle$

$\vec{B}|_{t=0} = \langle 0,3,-3\rangle$
$\vec{B}' = \langle \cos(t),-3e^{-t},3\sin(t)\rangle|_{t=0} = \langle 1,-3,0\rangle$
$\vec{B}'' = \langle -\sin(t),3e^{-t},3\cos(t)\rangle|_{t=0} = \langle 0,3,3\rangle$

$$\vec{A}''\times\vec{B}|_{t=0} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 0 & 2 \\ 0 & 3 & -3 \end{vmatrix} = \langle -6,18,18\rangle$$

$$\vec{A}'\times\vec{B}'|_{t=0} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & -2 \\ 1 & -3 & 0 \end{vmatrix} = \langle -6,-2,1\rangle$$

$$\vec{A}\times\vec{B}''|_{t=0} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -4 & 0 \\ 0 & 3 & 3 \end{vmatrix} = \langle -12,0,0\rangle$$

Therefore, $\frac{d^2}{dt^2}\left(\vec{A}\times\vec{B}\right)|_{t=0} = \langle -6,18,18\rangle + 2\langle -6,-2,1\rangle + \langle -12,0,0\rangle = \langle -30,14,20\rangle$

2. [20 points] The acceleration $\vec{a}$ of a particle at any time $t \geq 0$ is given by $\vec{a} = \langle e^{-t}, -6(t+1), 3\sin(t)\rangle$. If the velocity $\vec{v}$ and the displacement $\vec{r}$ are zero at $t=0$, find $\vec{v}$ and $\vec{r}$ at any time t.

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle -e^{-t}, -6(\frac{t^2}{2}+t), -3\cos(t)\rangle + \vec{C}$.
Since $\vec{v}(0) = \langle -1,0,-3\rangle + \vec{C} = \langle 0,0,0\rangle \to \vec{C} = \langle 1,0,3\rangle$.
So, $\vec{v}(t) = \langle 1-e^{-t}, -3t^2-6t, 3-3\cos(t)\rangle$.

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle t+e^{-t}, -3\frac{t^3}{3}-6\frac{t^2}{2}, 3t-3\sin(t)\rangle + \vec{D}$.
Since $\vec{r}(0) = \langle 1,0,0\rangle + \vec{D} = \langle 0,0,0\rangle \to \vec{D} = \langle -1,0,0\rangle$.
So, $\vec{r}(t) = \langle t+e^{-t}-1, -t^3-3t^2, 3t-3\sin(t)\rangle$.

3. [20 points] Use the Chain Rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ of $z = x^2y^3$, where $x = s\cos(t)$ and $y = s\sin(t)$.

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2xy^3)(\cos(t)) + (3x^2y^2)(\sin(t))$.
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2xy^3)(-s\sin(t)) + (3x^2y^2)(s\cos(t))$.

4. [20 points] Suppose you are climbing a hill whose shape is given by the equation $f(x,y) = z = 1000 - \frac{5}{1000}x^2 - \frac{1}{100}y^2$, where x, y. and z are measured in meters, and you are standing at a point with coordinates $(60, 40, 966)$.

The positive x-axis points east and the positive y-axis points north. *a)* If you walk due northwest, will you start to ascend or descend? At what rate? *b)* In which direction is the slope largest? What is the rate of ascent in that direction?

a) Here, we are asked to find the directional derivative of f in the direction of $\vec{u} = \langle -1, 1 \rangle$ (northwest). Thus, $D_{\vec{u}} f(60, 40) = \nabla f(60, 40) \cdot \hat{u}$, where $\nabla f(60, 40)$ is the gradient of the function at $(60, 40)$ (our current location) and $\hat{u}$ is a unit vector in the direction of $\vec{u}$. So, since $\nabla f(x, y) = \langle -\frac{1}{100}x, -\frac{2}{100}y \rangle$, $\nabla f(60, 40) = \langle -\frac{6}{10}, -\frac{8}{10} \rangle$. The unit vector $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

Therefore, $D_{\vec{u}} f(60, 40) = \nabla f(60, 40) \cdot \hat{u} = \langle -\frac{6}{10}, -\frac{8}{10} \rangle \cdot \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{6}{10\sqrt{2}} - \frac{8}{10\sqrt{2}} = -\frac{2}{10\sqrt{2}} = -\frac{1}{5\sqrt{2}}$. Since the directional derivative is negative, we descend if we walk due northwest and the rate at which we descend is $\frac{1}{5\sqrt{2}}$ vertical meters per horizontal meter.

b) The direction of largest slope at $(60, 40)$ is the same as the gradient $\nabla f(60, 40) = \langle -\frac{6}{10}, -\frac{8}{10} \rangle$. The rate of ascent is equal to the length of the gradient, or $|\nabla f(60, 40)| = |\langle -\frac{6}{10}, -\frac{8}{10} \rangle| = \sqrt{\frac{36}{100} + \frac{64}{100}} = 1$.

5. [20 points] Find the local maximum and minimum values and saddle point(s) of the function $f(x, y) = x^3 - 12xy + 8y^3$.

$f_x(x, y) = 3x^2 - 12y$ and $f_y(x, y) = 24y^2 - 12x$. At a critical point $f_x(x, y) = f_y(x, y) = 0$, so if $3x^2 - 12y = 0$ (1) and $24y^2 - 12x = 0$ (2), from (1) we have that $x^2 = 4y$ (3) and from (2) we have that $y^2 = x/2$ (4). Squaring (3) and substituting (4) in the result we have $x^4 = 16y^2 = 16(x/2) = 8x$. This means that $x^4 - 8x = x(x^3 - 8) = 0$. The solutions of this last equation are $x = 0$ and $x = 2$. Substituting x back in (3) we have that $y = 0$ and $y = 1$. So the critical points are $(0, 0)$ and $(2, 1)$.

Now the Hessian of f is $H(f(x, y)) = \begin{bmatrix} 6x & -12 \\ -12 & 48y \end{bmatrix}$. Then, $|H(f(0, 0))| = \begin{vmatrix} 0 & -12 \\ -12 & 0 \end{vmatrix} = -144 < 0$. This means that $(0, 0)$ is a saddle point. $|H(f(2, 1))| = \begin{vmatrix} 12 & -12 \\ -12 & 48 \end{vmatrix} = 12(48) - 12(12) = 12(36) > 0$. And since $f_{xx}(2, 1) = 12 > 0$, then $f(2, 1) = -8$ is a local minimum.

6. [Bonus problem: 10 points] You are told that there is a function f whose partial derivatives are $\frac{\partial f}{\partial x} = x + 4y$ and $\frac{\partial f}{\partial y} = 3x - y$. Should you believe it? (EXPLAIN.)

No. The reason is that $f_{xy} = 4$ and $f_{yx} = 3$, but since they are continuous, then Clairaut's theorem must hold. In other words, f_{xy} should be equal to f_{yx}.