University of Delaware
Department of Mathematical Sciences

MATH-243 – Analytical Geometry and Calculus C
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Spring 2012

Exam III

Solution

May 7 & 8, 2012

Question	1	2	3	4	5	Bonus	Total
Points							

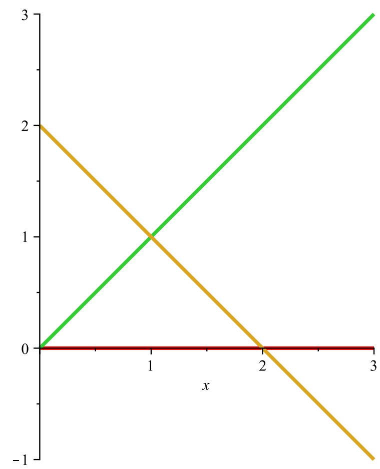
Instructions

- The exam is composed of **five** problems for a total of 100 points, plus a bonus problem for 10 extra points.
- Read very carefully each problem before working on it.
- Partial credit will not be given if appropriate work is not shown.
- If you get stuck on a problem, skip it and come back to it if you have extra time at the end.
- Answer questions in the space provided. If you need more space for an answer, continue your answer on the back of the page, or/and use the margins of the test pages.
- Carefully work out each problem and clearly indicate your final answer to any problem.
- You may **not** use calculators, dictionaries, notes, or any other kinds of aids.
- **DISHONESTY WILL NOT BE TOLERATED.**

Problems

1. [20 points] The average value of a function of two variables on a plane region D is defined as $f_{\text{avg}} = \dfrac{1}{A(D)} \displaystyle\iint_D f(x,y)\mathrm{d}A$, where $A(D)$ is the area of D. Using this information, calculate the average value of $f(x,y) = x - y^2$ over the region on the plane bounded by the lines $x + y = 2$, $x = y$, and $y = 0$.

   Solution: The region of integration is the isoceles triangle formed by the lines $y = 2 - x$, $y = x$, and $y = 0$ (See figure below).



   Based on the figure, we can say that

   $f_{\text{avg}} = \frac{1}{((\text{base})(\text{height}))/2} \iint_D (x - y^2)\mathrm{d}A = \frac{1}{((2)(1))/2} \int_0^1 \int_y^{2-y} (x - y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left[\frac{x^2}{2} - xy^2\right]_y^{2-y} \mathrm{d}y =$
   $\int_0^1 \left[(\frac{(2-y)^2}{2} - (2-y)y^2) - (\frac{y^2}{2} - y^3)\right] \mathrm{d}y = \int_0^1 (2 - 2y + \frac{y^2}{2} - 2y^2 + y^3 - \frac{y^2}{2} + y^3)\,\mathrm{d}y = \int_0^1 (2 - 2y - 2y^2 + 2y^3)\,\mathrm{d}y =$
   $\left[2y - y^2 - \frac{2y^3}{3} + \frac{y^4}{2}\right]_0^1 = 2 - 1 - \frac{2}{3} + \frac{1}{4} = \frac{7}{12}.$

2. [20 points] Evaluate $\displaystyle\oint_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F} = (x - 3y)\hat{i} + (y - 2x)\hat{j}$ and C is the closed curve in the xy-plane, $x = 2\cos(t)$, $y = 3\sin(t)$ from $t = 0$ to $t = 2\pi$.

   Solution 1: $\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$.
   $\vec{F}(\vec{r}(t)) = \langle 2\cos(t) - 3(3\sin(t)), 3\sin(t) - 2(2\cos(t))\rangle = \langle 2\cos(t) - 9\sin(t), 3\sin(t) - 4\cos(t)\rangle$.
   $\vec{r}\,'(t) = \langle -2\sin(t), 3\cos(t)\rangle$.

   Therefore $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 18\sin^2(t) - 12\cos^2(t) + 5\sin(t)\cos(t) = 18(1 - \cos^2(t)) - 12\cos^2(t) + 5\sin(t)\cos(t) = 18 - 30\cos^2(t) + 5\sin(t)\cos(t)$.

   Thus, $\oint_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} (18 - 30\cos^2(t) + 5\sin(t)\cos(t))\,dt$.

   Now,
   $\int_0^{2\pi} 18\,dt = 36\pi$

$30\int_0^{2\pi}\cos^2(t)\,dt =$ [By parts: $u' = \cos(t), v = \cos(t)$] $= 30\left[\frac{1}{2}(\sin(t)\cos(t)+t)\right]_0^{2\pi} = 30\pi$
$5\int_0^{2\pi}\sin(t)\cos(t)\,dt = 5\int_0^0 u\,du = 0$

We conclude then that
$\oint_C \vec{F}\cdot d\vec{r} = 36\pi - 30\pi = 6\pi$.

Solution 2: We can use Green's Theorem to find the value of the line integral. In this case, $P = x - 3y$ and $Q = y - 2x$, so $P_y = -3$, $Q_x = -2$. Therefore, since

$\oint_C \vec{F}\cdot \mathrm{d}\vec{r} = \iint_D (Qx - Py)\ \mathrm{d}A = \iint_D (-2+3)\ \mathrm{d}A = \iint_D \mathrm{d}A$, which means that $\oint_C \vec{F}\cdot \mathrm{d}\vec{r}$ is equal to the area of the region D enclosed by the curve C. In this case, D is an ellipse with semi-minor axis equal to 2 and a semi-major axis equal to 3, so its area is given by $(2)(3)\pi = 6\pi$.

3. [20 points] Determine whether or not $\vec{F} = (\ln(y) + 2xy^3)\hat{\imath} + (3x^2y^2 + x/y)\hat{\jmath}$ is a conservative vector field. If it is, find a function f such that $\vec{F} = \nabla f$. If it is not, explain why.

   Solution: $\vec{F}$ is defined only in $D = \{(x,y)|x \in \mathbb{R}, y > 0\}$. Since, D is simply connected, that is, it does not have any holes in it, $\vec{F}$ will be a conservative field if $Q_x = P_y$. In this case, $P = \ln(y) + 2xy^3$ and $Q = 3x^2y^2 + x/y$, so $P_y = \frac{1}{y} + 6xy^2$ and $Q_x = 6xy^2 + \frac{1}{y}$, which is defined in D. Therefore, we can conclude that $\vec{F}$ is a conservative field.

   To find f, such that $\vec{F} = \nabla f$, we integrate P with respect to x. When we do that, we obtain $f_{\text{candidate}} = x\ln(y) + x^2y^3 + C(y)$. Differentiating $f_{\text{candidate}}$ with respect to y we observe that $\frac{\partial f_{\text{candidate}}}{\partial y} = \frac{x}{y} + 3x^2y^2 + C_y(y)$. This means that $C_y(y) = 0$ and therefore that $C(y) = K$, where K is a constant.

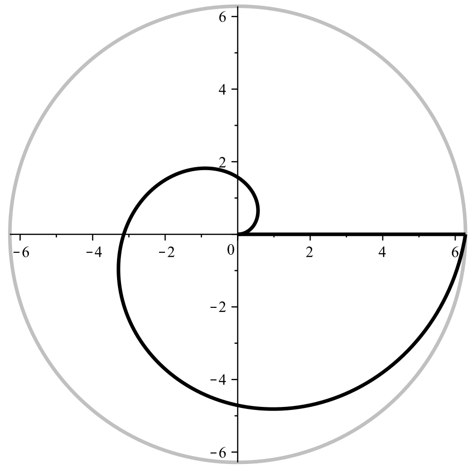
   We conclude that f for which $\vec{F} = \nabla f$ is equal to $x\ln(y) + x^2y^3 + K$.

4. [20 points] Use Green's Theorem to find the work done in moving a particle in the force field $\vec{F} = \langle x(x+y), xy^2\rangle$ from the origin along the x-axis to (1,0), then along a straight line segment to (0,1) and then back to the origin along the y-axis.

   Solution: By Green's Theorem, $\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$. In our case, $P = x(x+y)$ and $Q = xy^2$, which means that $Q_x = y^2$ and $P_y = x$. Therefore

   $\oint_C x(x+y)\,dx + xy^2\,dy = \iint_D (y^2 - x)\,dA = \int_0^1\int_0^{1-x}(y^2 - x)\,dy\,dx = \int_0^1\left[\frac{y^3}{3} - xy\right]_0^{1-x} dx =$
   $\int_0^1\left[\frac{(1-x)^3}{3} - x(1-x\right] dx = \frac{1}{3}\int_0^1 (1-x)^3\,dx - \int_0^1 x\,dx + \int_0^1 x^2\,dx = -\frac{1}{3}\int_1^0 u^3\,du - \frac{1}{2} + \frac{1}{3} = \frac{1}{3}\left(\frac{1}{4}\right) - \frac{1}{2} + \frac{1}{3} = -\frac{1}{12}$

5. [20 points] The figure below shows an Archimedes' spiral whose equation in polar coordinates is $r = \theta$, and a circle with equation in polar coordinates $r = 2\pi$. Using double integrals in polar coordinates, show that the area bounded by the spiral and the positive x-axis is equal to one third of the area bounded by the circle.



Solution: The area bounded by the circle is $A_c = \pi r^2$, but since $r = 2\pi$, $A_{\text{circle}} = \pi(2\pi)^2 = 4\pi^3$. Thus, we would like to show that $A_{\text{spiral}} = \iint_S \mathrm{d}A = \frac{4}{3}\pi^3$, where S is the region bounded by the Archimedes' spiral and the line segment from the origin to $(2\pi, 0)$. This area is in fact given by $A_{\text{spiral}} = \int_0^{2\pi}\int_0^{\theta} r\,\mathrm{d}r\,\mathrm{d}\theta$ because as the angle θ varies, so does the radius (by the definition of the spiral).

$\int_0^{2\pi}\int_0^{\theta} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{\theta}\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi}\theta^2\,\mathrm{d}\theta = \frac{1}{2}\left[\frac{\theta^3}{3}\right]_0^{2\pi} = \frac{1}{6}(8\pi^3) = \frac{4}{3}\pi^3$, which is what we wanted to show.

[Bonus problem: 10 points] Use the transformation $u = xy$ and $v = xy^2$ to evaluate $\iint_R y^2\mathrm{d}A$, where R is the region bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$, $xy^2 = 2$.

Solution: With the given transformation, $y^2 = (v/u)^2$ and the Jacobian of the transformation is $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} y & x \\ y^2 & 2xy \end{vmatrix} = 2xy^2 - xy^2 = xy^2 = v$, which means that $\mathrm{d}u\mathrm{d}v = |v|\mathrm{d}x\mathrm{d}y$, and therefore $\mathrm{d}x\mathrm{d}y = \frac{1}{v}\mathrm{d}u\mathrm{d}v$, since $v > 0$.

So, $\iint_R y^2\mathrm{d}A = \int_1^2\int_1^2 (\frac{v}{u})^2(\frac{1}{v})\,\mathrm{d}u\mathrm{d}v = \int_1^2\int_1^2 \frac{v}{u^2}\,\mathrm{d}u\mathrm{d}v = \int_1^2 v\left[-u^{-1}\right]_1^2\mathrm{d}v = \int_1^2 v\left[-2^{-1}+1\right]\mathrm{d}v = \frac{1}{2}\int_1^2 v\,\mathrm{d}v = \frac{1}{2}\left[\frac{v^2}{2}\right]_1^2 = \frac{1}{2}\left(\frac{4}{2} - \frac{1}{2}\right) = \frac{3}{4}$.