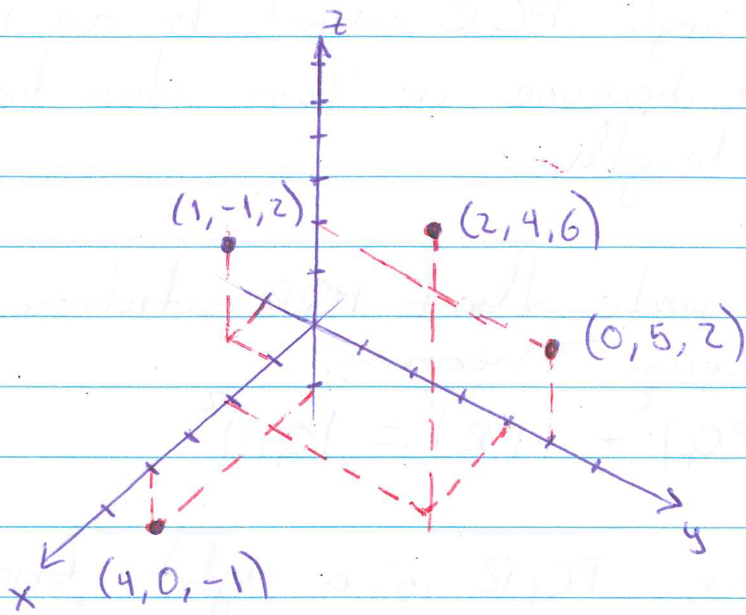
Homework #1: Solutions

1.



2. We need to compute the distance from P to Q, from P to R, and from Q to R.

$$|PQ| = \sqrt{(4-2)^2 + (1-(-1))^2 + (1-0)^2}$$
$$= \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$|PR| = \sqrt{(4-2)^2 + (-5-(-1))^2 + (4-0)^2}$$
$$= \sqrt{(2)^2 + (-4)^2 + (4)^2} = \sqrt{4+16+16} = \sqrt{36} = 6$$

$$|QR| = \sqrt{(4-4)^2 + (-5-1)^2 + (4-1)^2} = \sqrt{0 + (-6)^2 + (3)^2} =$$
$$= \sqrt{36+9} = \sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}$$

The triangle PQR cannot be an isosceles triangle because no two sides have the same length.

Now, note that PQR satisfies the Pythagorean Theorem:

$$|PQ|^2 + |PR|^2 = |QR|^2$$

Therefore, PQR is a right triangle.

3. We can rewrite

$$x^2+y^2+z^2-2x-4y+8z = 15$$

as follows:

$$(x^2-2x)+(y^2-4y)+(z^2+8z)=15$$

Completing squares:

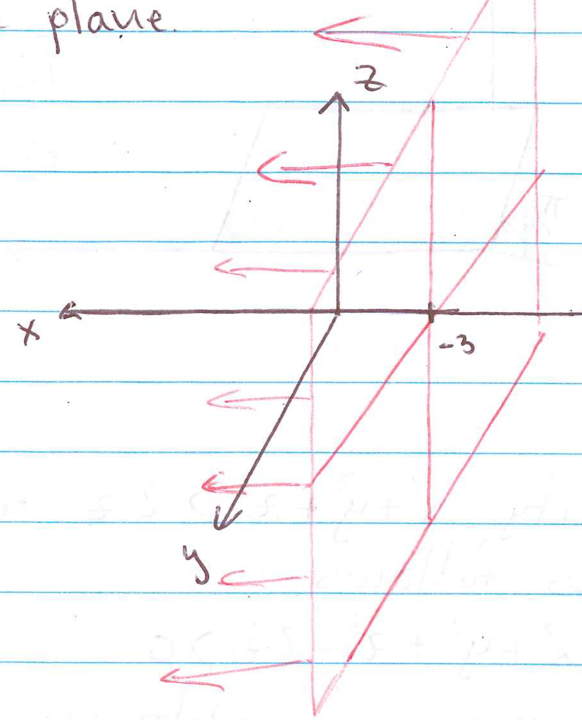
$$(x^2-2x+1)+(y^2-4y+4)+(z^2+8z+16)=15+1+4+16$$
$$=36$$

Thus

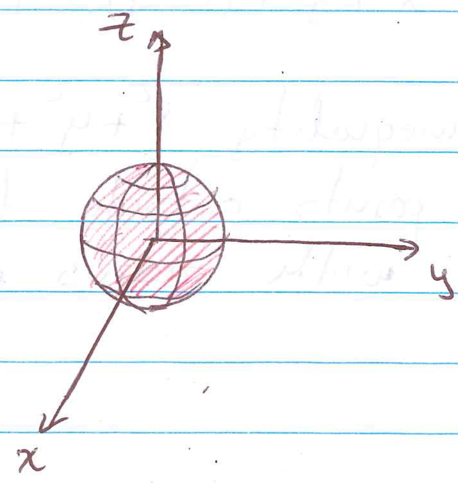
$$(x-1)^2+(y-2)^2+(z+4)^2=36$$

$\therefore$ The original equation describes a sphere centered at $(1,2,-4)$ with radius $\sqrt{36}=6$.

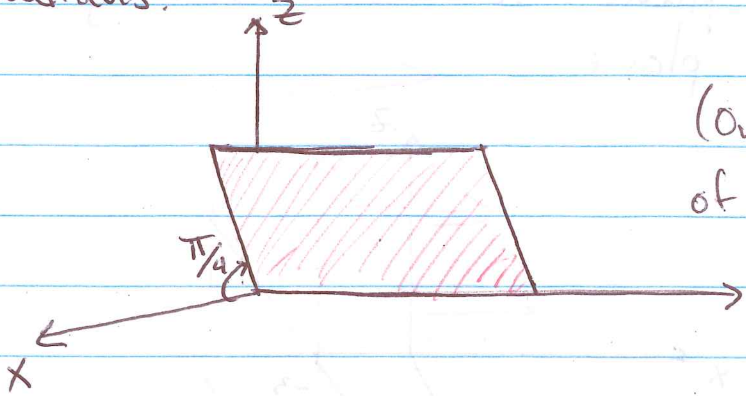
4. a) The inequality $x \geq -3$ represents a half-space consisting of all points in front of the plane $x = -3$, including the points on the plane.



b) The inequality $x^2 + y^2 + z^2 \leq 3$ represents the set of all points on or inside the sphere with radius $\sqrt{3}$ and center $(0,0,0)$.



c) The equation $x = z$ represents the plane that bisects the xz-plane at an angle of $\pi/4$ radians.



(Only a portion of the plane in the 1st octant is shown.)

d) The inequality $x^2+y^2+z^2 > 2z$ can be rewritten as follows:

$$x^2+y^2+z^2-2z > 0$$

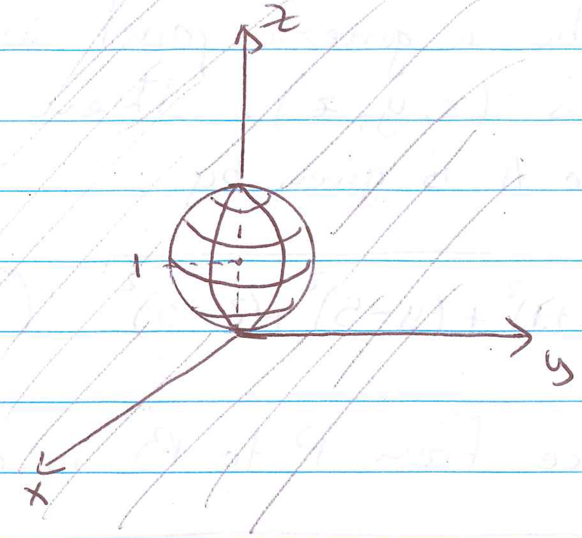
Completing squares we obtain

$$x^2+y^2+(z^2-2z+1) > 1$$

which can be written as follows

$$x^2+y^2+(z-1)^2 > 1$$

Thus, the inequality $x^2+y^2+z^2 > 2z$ represents the set of points outside the shere centered at $(0,0,1)$ with radius equal to 1.



5.- A solid sphere of radius 2 centered at the origin is described by

$$x^2 + y^2 + z^2 \leq 4$$

However, we need only the upper hemishere which corresponds to values of $z \geq 0$.

Therefore, the pair of inequalities that represent the upper hemisphere of a solid sphere of radius 2 centered at the origin are

$$x^2 + y^2 + z^2 \leq 4 \ , \quad z \geq 0$$

6. Let P be a generic point with coordinates (x, y, z). Then the distance from P to A is given by

$$|PA| = \sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} \quad (1)$$

The distance from P to B is given by

$$|PB| = \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2} \quad (2)$$

The relationship between $|PA|$ and $|PB|$ is given by

$$|PA| = 2|PB| \quad (3)$$

Squaring both sides of (3) we get

$$|PA|^2 = 4|PB|^2 \quad (4)$$

Substituting (1) and (2) in (4) we obtain

$$\left(\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2}\right)^2 = 4\left(\sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2}\right)^2$$

$$(x+1)^2 + (y-5)^2 + (z-3)^2 = 4\left[(x-6)^2 + (y-2)^2 + (z+2)^2\right]$$

Expanding and rearranging we get

$$x^2+2x+1+y^2-10y+25+z^2-6z+9 = 4(x^2-12x+36+y^2-4y+4+z^2+4z+4)$$

$$x^2+y^2+z^2+2x-10y-6z+35 = 4(x^2+y^2+z^2-12x-4y+4z+44)$$

$$x^2+y^2+z^2+2x-10y-6z+35 = 4x^2+4y^2+4z^2-48x-16y+16z+176$$

$$4x^2-x^2+4y^2-y^2+4z^2-z^2-48x-2x-16y+10y+16z+6z+176-35=0$$

$$3x^2+3y^2+3z^2-50x-6y+22z+141=0$$

Dividing by 3 we get

$$x^2+y^2+z^2-\frac{50}{3}x-2y+\frac{22}{3}z+47=0$$

Rearranging and completing squares:

$$\left(x^2-\frac{50}{3}x\right)+\left(y^2-2y\right)+\left(z^2+\frac{22}{3}z\right)=-47$$

$$\left(x^2-\frac{50}{3}x+\frac{(25)^2}{(3)^2}\right)+\left(y^2-2y+1\right)+\left(z^2+\frac{22}{3}z+\frac{(11)^2}{(3)^2}\right)=$$

$$-47+1+\frac{25^2}{3^2}+\frac{11^2}{3^2}$$

$$\left(x-\frac{25}{3}\right)^2+\left(y-1\right)^2+\left(z+\frac{11}{3}\right)^2=\frac{332}{3^2}$$

$\therefore$ All points P represent a sphere with radius $\frac{\sqrt{332}}{3}$ centered at $\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$

7. Let $P(x,y,z)$ represent the set of points that satisfy

$$|PA| = |PB|$$

So we have

$$\sqrt{(x+1)^2+(y-5)^2+(z-3)^2}=\sqrt{(x-6)^2+(y-2)^2+(z+2)^2}$$

Squaring both sides and expanding we obtain

$$(x+1)^2+(y-5)^2+(z-3)^2=(x-6)^2+(y-2)^2+(z+2)^2$$

$$\cancel{x^2}+2x+1+\cancel{y^2}-10y+25+\cancel{z^2}-6z+9=\cancel{x^2}-12x+36+\cancel{y^2}-4y+4+\cancel{z^2}+4z+4$$

$$2x-10y-6z+35=-12x-4y+4z+44$$

$$2x+12x-10y+4y-6z-4z+35-44=0$$

$$14x-6y-10z-9=0$$ ← This represents a plane that bisects the line segment AB.