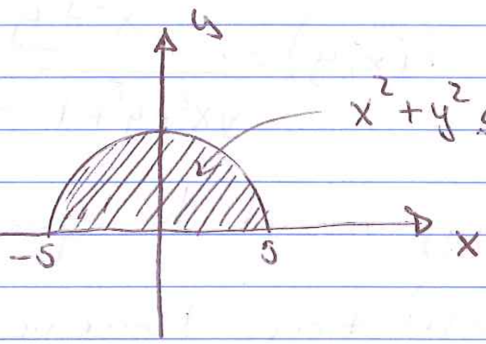
Homeworks 10 & 11

1.a) $\sqrt{y} + \sqrt{25-x^2-y^2}$ is defined only when

$$y \geq 0 \quad \text{and} \quad 25 - x^2 - y^2 \geq 0$$
$$(1) \qquad\qquad (2)$$

From (2): $x^2 + y^2 \leq 25 = (5)^2$

Thus, the domain of $\sqrt{y} + \sqrt{25-x^2-y^2}$ is a half disk of radius 5.



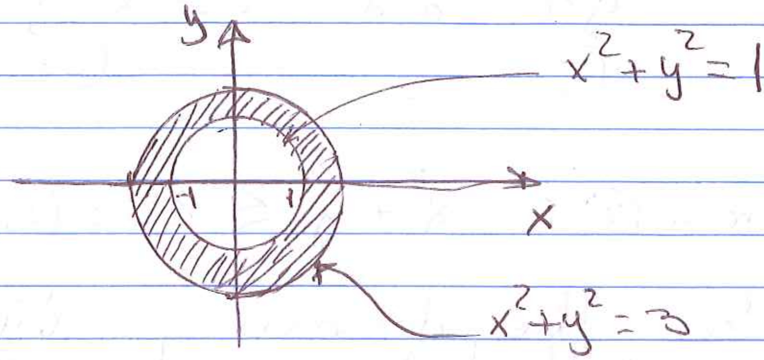
1.b) The arcsin function is defined only when its argument is between -1 and 1. So

$$-1 \leq x^2 + y^2 - 2 \leq 1$$

Adding 2:

$$1 \leq x^2 + y^2 \leq 3$$

This means that the domain of $\arcsin(x^2+y^2-2)$ is the set of points that are outside the circle $x^2+y^2=1$ and inside the disk $x^2+y^2 \leq 3$. Graphically:



2. a) Let $f(x,y) = \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

$\lim_{(x,y)\to(0,0)} f(x,y)$ would be equal to $\frac{0}{0}$ by direct substitution. However if we rationalize the denominator we obtain:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} =$$

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\cdot(\sqrt{x^2+y^2+1}+1)}{(x^2+y^2+1)-1} =$$

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{x^2+y^2} =$$

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1}+1 = 2$$

2.b) Let $f(x,y,z) = \dfrac{yz}{x^2+4y^2+9z^2}$

If we approach the origin along any axis, that is, if we approach $f(x,0,0)$ or $f(0,y,0)$ or $f(0,0,z)$ as $x \to 0$, $y \to 0$, or $z \to 0$, respectively $f(x,y,z) \to 0$. However if we approach the origin using the path represented by $y=z$, $x=0$ we have:

$$f(0,y,y) = \frac{y^2}{4y^2+9y^2} = \frac{y^2}{13y^2} = \frac{1}{13}$$

Therefore

$$\lim_{(x,y,z)\to(0,0,0)} \frac{1}{13} = \frac{1}{13}$$ and this means that the limit $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$ does not exist.

3. Let $f(x,t) = \sqrt{x}\ln(t)$, then

$$\frac{\partial f}{\partial x} = \ln(t)\left(\frac{1}{2\sqrt{x}}\right) = \frac{\ln(t)}{2\sqrt{x}}$$

$$\frac{\partial f}{\partial t} = \sqrt{x}\left(\frac{1}{t}\right) = \frac{\sqrt{x}}{t}$$

4. $$w = \frac{e^v}{u + v^2}$$

$$\frac{\partial w}{\partial v} = \frac{(u+v^2)e^v - e^v(2v)}{(u+v^2)^2} = \frac{e^v(u+v^2-2v)}{(u+v^2)^2}$$

$$\frac{\partial w}{\partial u} = e^v \frac{\partial}{\partial u}\left(\frac{1}{u+v^2}\right) = e^v\left(\frac{(u+v^2)\emptyset - 1(1)}{(u+v^2)^2}\right) =$$

$$= \frac{-e^v}{(u+v^2)^2}$$

5. Given that $f(x,y) = \int_y^x \cos(t^2)\,dt$ and that we want to find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, we may use the fundamental theorem of calculus in each case because in order to find the partial derivatives of f we hold one of the variables constant. Therefore:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\int_y^x \cos(t^2)\,dt\right) = \cos(x^2) \text{ by the FTC.}$$

in this case y is constant.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int_y^x \cos(t^2)\,dt\right) = \frac{\partial}{\partial y}\left(-\int_{x}^{y} \cos(t^2)\,dt\right) \Rightarrow$$

now x is constant

$$\frac{\partial f}{\partial y} = -\cos(y^2)$$

6. $v = \sin\left(\sum_{i=1}^{n} i x_i\right)$

Since there are n variables, we have n first order partial derivatives. Let us find the ith partial derivative:

$$\frac{\partial v}{\partial x_i} = \cos\left(\sum_{i=1}^{n} i x_i\right)(i)$$

So

$$\frac{\partial v}{\partial x_1} = \cos\left(\sum_{i=1}^{n} i x_i\right)$$

$$\frac{\partial v}{\partial x_2} = \cos\left(\sum_{i=1}^{n} i x_i\right)(2) = 2\cos\left(\sum_{i=1}^{n} i x_i\right)$$

$$\frac{\partial v}{\partial x_3} = \cos\left(\sum_{i=1}^{n} i x_i\right)(3) = 3\cos\left(\sum_{i=1}^{n} i x_i\right)$$

$$\vdots$$

$$\frac{\partial v}{\partial x_n} = \cos\left(\sum_{i=1}^{n} i x_i\right)(n) = n\cos\left(\sum_{i=1}^{n} i x_i\right)$$

7. Given $yz = \ln(x+z)$ we differentiate both sides of the equation:

$$\frac{\partial}{\partial x}(yz) = \frac{\partial}{\partial x}\left(\ln(x+z)\right)$$

$$\frac{\partial y}{\partial x} z + y\frac{\partial z}{\partial x} = \frac{1}{(x+z)}\frac{\partial}{\partial x}(x+z) = \frac{1+\frac{\partial z}{\partial x}}{x+z}$$

if we assume that y is independent of x $\frac{\partial y}{\partial x} = 0$, thus

$$y\frac{\partial z}{\partial x} = \frac{1}{x+z} + \frac{\frac{\partial z}{\partial x}}{x+z}$$

$$y\frac{\partial z}{\partial x} - \frac{\frac{\partial z}{\partial x}}{x+z} = \frac{1}{x+z}$$

$$\frac{\partial z}{\partial x}\left(y - \frac{1}{x+z}\right) = \frac{1}{x+z}$$

$$\frac{\partial z}{\partial x}\left(\frac{y(x+z)-1}{x+z}\right) = \frac{1}{x+z}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{1}{y(x+z)-1}}$$

To find $\frac{\partial z}{\partial y}$ we follow the same steps:

$$\frac{\partial}{\partial y}(yz) = \frac{\partial}{\partial y}\left(\ln(x+z)\right)$$

$$\frac{\partial y}{\partial y} z + y\frac{\partial z}{\partial y} = \frac{1}{x+z}\left(\frac{\partial}{\partial y}(x+z)\right)$$

$$z + y\frac{\partial z}{\partial y} = \frac{1}{x+z}\left(\frac{\partial z}{\partial y}\right)$$

$$z = \frac{1}{x+z}\left(\frac{\partial z}{\partial y}\right) - y\left(\frac{\partial z}{\partial y}\right) = \frac{1-y(x+z)}{x+z}\left(\frac{\partial z}{\partial y}\right)$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{z(x+z)}{1-y(x+z)}}$$

8. $z = \arctan\left(\frac{x+y}{1-xy}\right)$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\cdot\left[\frac{(1-xy)\frac{\partial}{\partial x}(x+y) - (x+y)\frac{\partial}{\partial x}(1-xy)}{(1-xy)^2}\right]$$

$$= \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\cdot\left[\frac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2}\right]$$

$$\frac{\partial z}{\partial x} = \frac{1}{\frac{(1-xy)^2+(x+y)^2}{(1-xy)^2}} \cdot \left[\frac{1-xy+y(x+y)}{(1-xy)^2}\right]$$

$$= \frac{\cancel{(1-xy)^2}}{(1-xy)^2+(x+y)^2}\left[\frac{1+y^2}{\cancel{(1-xy)^2}}\right]$$

$$= \frac{1+y^2}{1-\cancel{2xy}+x^2y^2+x^2+\cancel{2xy}+y^2} = \frac{\cancel{1+y^2}}{\cancel{(1+y^2)}(1+x^2)} =$$

$$= \frac{1}{1+x^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\left[\frac{(1-xy)\frac{\partial}{\partial y}(x+y)-(x+y)\frac{\partial}{\partial y}(1-xy)}{(1-xy)^2}\right]$$

$$= \frac{\cancel{(1-xy)^2}}{(1-xy)^2+(x+y)^2}\left[\frac{(1-xy)(1)-(x+y)(-x)}{\cancel{(1-xy)^2}}\right]$$

$$= \frac{1+x^2}{(1+y^2)(1+x^2)} = \frac{1}{1+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left((1+x^2)^{-1}\right) = -1(1+x^2)^{-2}(2x)$$

$$= \boxed{-\frac{2x}{(1+x^2)^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left((1+y^2)^{-1}\right) = -1(1+y^2)^{-2}(2y)$$

$$= \boxed{-\frac{2y}{(1+y^2)^2}}$$

$$\boxed{\frac{\partial^2 z}{\partial y \partial x} = 0 \quad ; \quad \frac{\partial^2 z}{\partial x \partial y} = 0}$$

9. a) Fixing the value of y and going through P in the positive x-direction, we see that f decreases. Therefore, f_x is negative at P.

b) If we fix x and go through P in the positive y-direction we see that f increases. Therefore, f_y is positive at P.

c) f_{xx} is the rate of change of f_x as x increases. In the figure it is apparent that as we go through P from left to right the rate at which the function decreases is slowing down (the distance between the

level curve $f=8$ and the level curve $f=6$ is smaller than the distance between the level curve $f=6$ and the level curve $f=4$).
Therefore, f_{xx} is positive.

d) [This one is difficult to answer because the figure is not very clear.]
f_{xy} represents the rate of change of f_x as y increases. Thus, you should imagine how the slope of the tangent line parallel to x at P changes as y increases. Toward the top of the figure you see that the level curves are further away from each other than they are at P. Thus, we can conclude that the rate of change wrt x decreases as y increases.
Therefore $f_{xy} > 0$.

e) f_{yy} is the rate of change of f_y as y increases. Since the distance between the level curves decreases as we move through P in the positive y-direction, the rate of change of f_y increases. Thus $f_{yy} > 0$.

10. $z = \ln(e^x + e^y)$

$$\frac{\partial z}{\partial x} = \frac{1}{e^x + e^y} e^x \ ; \ \frac{\partial z}{\partial y} = \frac{1}{e^x + e^y} e^y$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{e^x}{e^x+e^y} + \frac{e^y}{e^x+e^y} = \frac{e^x+e^y}{e^x+e^y} = 1$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{(e^x+e^y)e^x - e^x(e^x)}{(e^x+e^y)^2} = \frac{e^{x+y}}{(e^x+e^y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(e^x+e^y)e^y - e^y(e^y)}{(e^x+e^y)^2} = \frac{e^{x+y}}{(e^x+e^y)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(e^x+e^y)(0) - e^y(e^x)}{(e^x+e^y)^2} = -\frac{e^{x+y}}{(e^x+e^y)^2}$$

So:

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \frac{e^{x+y}}{(e^x+e^y)^2} \cdot \frac{e^{x+y}}{(e^x+e^y)^2} - \left(-\frac{e^{x+y}}{(e^x+e^y)^2}\right)^2 =$$

$$= \frac{(e^{x+y})^2}{(e^x+e^y)^4} - \frac{(e^{x+y})^2}{(e^x+e^y)^4} = 0$$

11. $z = y\ln(x)$ @ $(1,4,0)$

Eq. of tangent plane:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Thus

$$f_x(x,y) = y\frac{1}{x} = \frac{y}{x}\ ;\ f_x(1,4) = \frac{4}{1} = 4$$

$$f_y(x,y) = \ln(x)\ ;\ f_y(1,4) = \ln(1) = 0$$

So:

$$z - 0 = 4(x-1) + 0(y-4) \Rightarrow \boxed{z = 4(x-1)}$$

12. $z = e^{x^2-y^2} \Rightarrow \frac{\partial z}{\partial x} = \left(e^{x^2-y^2}\right)(2x)$ and

$\frac{\partial z}{\partial y} = \left(e^{x^2-y^2}\right)(-2y)$. So

$$f_x(1,-1) = \left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\y=-1}} = \left(e^{1-1}\right)(2(1)) = 2$$

$$f_y(1,-1) = \left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=-1}} = \left(e^{1-1}\right)(-2(-1)) = 2$$

Now, the tangent plane is given by

$$\boxed{z - 1 = 2(x-1) + 2(y+1)}$$

13. $f(x,y) = \ln(x-3y)$

The linearization of $f(x,y)$ at $(7,2)$ is:

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

So we need to find f_x and f_y.

$f_x(x,y) = \dfrac{1}{x-3y}$; $f_y(x,y) = \dfrac{-3}{x-3y}$

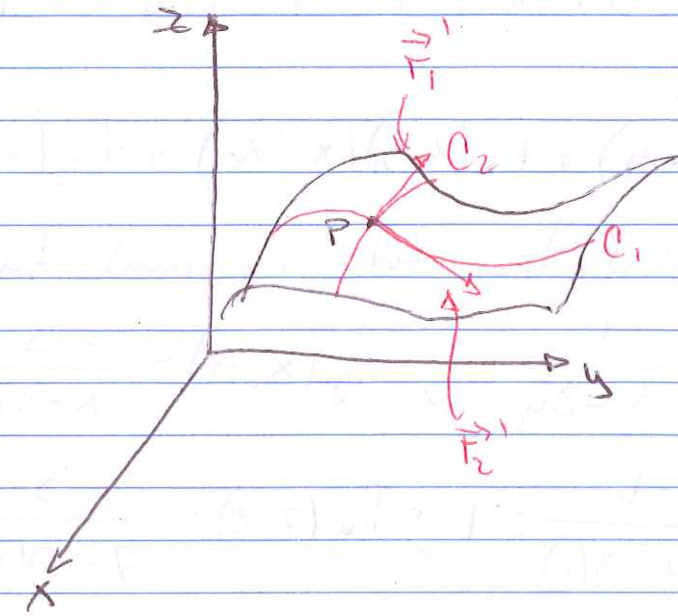
$f_x(7,2) = \dfrac{1}{7-3(2)} = 1$; $f_y(7,2) = \dfrac{-3}{7-3(2)} = -3$

$$L(x,y) = \ln(7-3(2)) + 1(x-7) + (-3)(y-2)$$
$$= \ln(1) + (x-7) - 3(y-2)$$
$$= x - 7 - 3y + 6 = \boxed{x - 3y - 1}$$

Now $f(6.9, 2.06) \approx L(6.9, 2.06)$

$$\approx 6.9 - 3(2.06) - 1$$
$$\approx \boxed{-0.28}$$

14. Consider the following figure:



The vectors $\vec{r}_1'$ and $\vec{r}_2'$ are the derivatives of the spacecurves described by $\vec{r}_1(t) = \langle 2+3t, 1-t^2, 3-4t+t^2\rangle$ and $\vec{r}_2(u) = \langle 1+u^2, 2u^3-1, 2u+1\rangle$ at P.

So:

$\vec{r}_1'(t) = \langle 3, -2t, -4+2t\rangle$

$\vec{r}_2'(u) = \langle 2u, 6u^2, 2\rangle$

$\vec{r}_1(t)$ passes through P when $t=0$ and $\vec{r}_2(u)$ passes through P when $u=1$. So

$\vec{r}_1'(0) = \langle 3, 0, -4\rangle$ and $\vec{r}_2'(1) = \langle 2, 6, 2\rangle$.

A normal vector for the tangent plane is

$$\vec{n} = \vec{r}_1'(0) \times \vec{r}_2'(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -4 \\ 2 & 6 & 2 \end{vmatrix} = \langle 24, -14, 18\rangle.$$

So an equation of the tangent plane is:

$$24(x-2) - 14(y-1) + 18(z-3) = 0$$

or

$$12x - 7y + 9z = 44$$

15. $z = \arctan\left(\frac{y}{x}\right)$, $x = e^{t}$, $y = 1 - e^{-t}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} \cdot e^{t} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} \cdot e^{-t}$$

$$= -\frac{y}{x^2+y^2} \cdot e^{t} + \frac{1}{x+\frac{y^2}{x}} e^{-t} = \boxed{\frac{xe^{-t} - ye^{t}}{x^2+y^2}}$$

16.

$$\frac{\partial N}{\partial v} = \frac{\partial N}{\partial p} \cdot \frac{\partial p}{\partial v} + \frac{\partial N}{\partial q} \cdot \frac{\partial q}{\partial v} + \frac{\partial N}{\partial r} \cdot \frac{\partial r}{\partial v}$$

$$\frac{\partial N}{\partial p} = \frac{(p+r)(1) - (p+q)(1)}{(p+r)^2} = \frac{p+r-p-q}{(p+r)^2} = \frac{r-q}{(p+r)^2}$$

$$\frac{\partial N}{\partial q} = \frac{1}{p+r}\left(\frac{\partial}{\partial q}(p+q)\right) = \frac{1}{p+r}$$

$$\frac{\partial N}{\partial r} = p+q\left(\frac{\partial}{\partial r}\left(\frac{1}{p+r}\right)\right) = (p+q)(-1)(p+r)^{-2}(1)$$

$$= \frac{-(p+q)}{(p+r)^2}$$

$$\frac{\partial p}{\partial v} = w \ ; \ \frac{\partial q}{\partial v} = 1 \ ; \ \frac{\partial r}{\partial v} = u$$

Thus

$$\frac{\partial N}{\partial v} = \frac{r-q}{(p+r)^2} \cdot w + \frac{1}{p+r} \cdot 1 - \frac{p+q}{(p+r)^2} \cdot u$$

$$= \frac{w(r-q) + (p+r) - u(p+q)}{(p+r)^2}$$

Now, since $r = 4 + 2(3) = 10$

$q = 3 + 2(4) = 11$

$p = 2 + 3(4) = 14$

$$\left.\frac{\partial N}{\partial v}\right|_{\substack{u=2 \\ v=3 \\ w=4}} = \frac{4(10-11) + (14+10) - 2(14+11)}{(14+10)^2}$$

$$= \frac{-4+24-50}{(24)^2} = \frac{-30}{(24)^2} = \frac{-5}{96}$$

17. $f(x,y) = y^2/x$

a) $\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle -\frac{y^2}{x^2}, \frac{2y}{x} \right\rangle$

b) $\nabla f(1,2) = \langle -4, 4 \rangle$

c) If $\vec{v} = \frac{1}{3}\langle 2, \sqrt{5} \rangle$, then

$$D_{\vec{v}} f(1,2) = \nabla f(1,2) \cdot \vec{v} = \langle -4, 4 \rangle \cdot \left\langle \frac{2}{3}, \frac{\sqrt{5}}{3} \right\rangle$$

$$= -\frac{8}{3} + \frac{4\sqrt{5}}{3} = \frac{4}{3}(\sqrt{5}-2)$$

18. $f(x,y) = \sin(xy)$

$\nabla f(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$

$\nabla f(1,0) = \langle 0, 1 \rangle$. Thus the maximum rate of change is $|\nabla f(1,0)| = 1$ and occurs in the direction $\langle 0, 1 \rangle$.

19. Since the direction of fastest change is $\nabla f(x,y) = \langle 2x-2, 2y-4 \rangle$, we need to find all points (x,y) so that

$$\nabla f(x,y) = \alpha \langle 1, 1 \rangle \quad \text{(parallel vectors)}$$

$$\langle 2x-2, 2y-4 \rangle = \langle \alpha, \alpha \rangle$$

Thus $2x-2 = \alpha = 2y-4 \Rightarrow 2x-2 = 2y-4$

So $x-1=y-2 \Rightarrow y=x-1+2=x+1$

So, the direction of fastest change is $\langle 1,1\rangle$ at all the points on the line given by $\underline{y=x+1}$.

20.

a) $$\nabla(au+bv)=\left\langle \frac{\partial}{\partial x}(au+bv), \frac{\partial}{\partial y}(au+bv)\right\rangle$$
$$=\left\langle a\frac{\partial u}{\partial x}+b\frac{\partial v}{\partial x}, a\frac{\partial u}{\partial y}+b\frac{\partial v}{\partial y}\right\rangle$$
$$=a\left\langle\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle+b\left\langle\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right\rangle$$
$$=a\nabla u+b\nabla v$$

b) $$\nabla(uv)=\left\langle v\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial x}, v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}\right\rangle$$
$$=v\left\langle\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle+u\left\langle\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right\rangle$$
$$=v\nabla u+u\nabla v$$

c) $$\nabla\left(\frac{u}{v}\right) = \left\langle \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}, \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2} \right\rangle$$

$$= \frac{1}{v^2}\left[v\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle - u\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle \right]$$

$$= \boxed{\frac{v\nabla u - u\nabla v}{v^2}}$$

d) $$\nabla u^n = \left\langle \frac{\partial}{\partial x} u^n, \frac{\partial}{\partial y} u^n \right\rangle$$

$$= \left\langle n u^{n-1}\frac{\partial u}{\partial x}, n u^{n-1}\frac{\partial u}{\partial y} \right\rangle$$

$$= \boxed{n u^{n-1} \nabla u}$$