Homework #12

1. Optimize $f(x,y) = 4x + 6y$ subject to $x^2+y^2=13$.

The Lagrange multiplier equation is

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

where $\nabla f(x,y) = \langle 4,6 \rangle$ and $\nabla g(x,y) = \langle 2x, 2y \rangle$

Then $4 = 2\lambda x$ and $6 = 2\lambda y$. Substituting x and y in the original constraint, we obtain:

$$x^2+y^2=13 \Rightarrow \left(\frac{2}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 = 13 \Rightarrow 4+9 = 13\lambda^2 \Rightarrow \lambda = \pm 1.$$

Therefore, candidate points are $(2,3)$ and $(-2,-3)$. Evaluating the function at those points we observe that $f(2,3)=26$ and $f(-2,-3)=-26$.

We can conclude then that $f(2,3)$ is a maximum and $f(-2,-3)$ is a minimum.

② Optimize $f(x,y,z) = 3x - y - 3z$ subject to $x+y-z=0$ and $x^2+2z^2=1$

We proceed as in the previous example, except for the fact that we now have two Lagrange multipliers:

$$\nabla f(x,y,z) = \lambda_1 \nabla g(x,y,z) + \lambda_2 \nabla h(x,y,z)$$
$$\langle 3,-1,-3\rangle = \lambda_1 \langle 1,1,-1\rangle + \lambda_2 \langle 2x, 0, 4z\rangle$$
$$= \langle \lambda_1 + 2\lambda_2 x,\ \lambda_1,\ -\lambda_1 + 4\lambda_2 z\rangle$$

$\therefore\ 3 = \lambda_2 + 2\lambda_2 x$	Given that $\lambda_1 = -1$
$-1 = \lambda_1$	
$-3 = -\lambda_1 + 4\lambda_2 z$	$4 = 2\lambda_2 x \Rightarrow x = \dfrac{2}{\lambda_2}$
$x+y-z=0$	
$x^2+2z^2=1$	$-4 = 4\lambda_2 z \Rightarrow z = \dfrac{-1}{\lambda_2}$

Substituting x and z in $x^2+2z^2=1$:

$$\frac{4}{\lambda_2^2} + 2\frac{1}{\lambda_2^2} = 1 \Rightarrow 6 = \lambda_2^2 \Rightarrow \lambda_2 = \pm\sqrt{6}$$

This means that $x = \pm\dfrac{2}{\sqrt{6}}$, $z = \mp\dfrac{1}{\sqrt{6}}$ and $y = z - x$ (from $x+y-z=0$) $\Rightarrow y = \mp\dfrac{3}{\sqrt{6}}$

The candidate points are therefore: $\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$ and $\left(\frac{-2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$.

Evaluating:

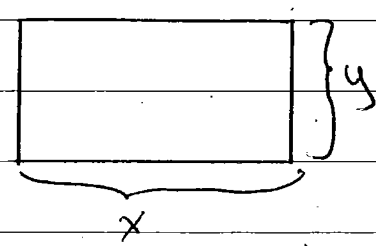
$$f\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = \frac{6+3+3}{\sqrt{6}} = \frac{12}{\sqrt{6}} = 2\sqrt{6}$$

$$f\left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = \frac{-6-3-3}{\sqrt{6}} = \frac{-12}{\sqrt{6}} = -2\sqrt{6}$$

So $f\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$ is a maximum

and $f\left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ is a minimum.

3) The area of a rectangle whose sides are x and y (as in the figure) is xy.



The perimeter of this rectangle is $2x+2y=p$.

To find the dimensions of the rectangle of largest area with a perimeter p, we need to

$$\text{maximize } f(x,y) = xy$$
$$\text{subject to } 2x + 2y = p.$$

Using a Lagrange multiplier we have

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$\langle y, x \rangle = \lambda \langle 2, 2 \rangle$$

Therefore

$$y = 2\lambda \quad (1)$$
$$x = 2\lambda \quad (2)$$
$$2x + 2y = p \quad (3)$$

x and y in (3):

$$4\lambda + 4\lambda = p \Rightarrow \lambda = \frac{p}{8}$$

Back in (1) and (2):

$$y = 2\left(\frac{p}{8}\right) = \frac{p}{4} \quad \text{and} \quad x = 2\left(\frac{p}{8}\right) = \frac{p}{4}$$

Since $x = y$, the rectangle of largest area is a square whose sides have length $\frac{p}{4}$.

4) The distance from any point (x, y, z) to the origin is $d = \sqrt{x^2+y^2+z^2}$. However, minimizing d is the same as minimizing d^2. Therefore, the function $f(x,y,z)$ that we will be maximizing/minimizing is $f(x,y,z) = d^2 = x^2+y^2+z^2$ subject to the constraints $x+y+2z=2$ and $z=x^2+y^2$ which forces the points to be part of the cylinder $z=x^2+y^2$ and the plane $x+y+2z=2$.

So, our problem can be stated as

$$
\begin{aligned}
\text{max/min} \quad & x^2+y^2+z^2 \\
\text{subject to} \quad & x+y+2z-2=0 \\
& x^2+y^2-z=0
\end{aligned}
$$

The Lagrange multiplier equation is

$$
\begin{aligned}
\nabla f(x,y,z) &= \lambda_1 \nabla g(x,y,z) + \lambda_2 \nabla h(x,y,z) \\
\langle 2x, 2y, 2z \rangle &= \lambda_1 \langle 1, 1, 2 \rangle + \lambda_2 \langle 2x, 2y, -1 \rangle \\
&= \langle \lambda_1 + 2\lambda_2 x, \lambda_1 + 2\lambda_2 y, 2\lambda_1 - \lambda_2 \rangle
\end{aligned}
$$

Therefore, the system of equations we need to solve is

$$2x = \lambda_1 + 2\lambda_2 x \quad (1)$$
$$2y = \lambda_1 + 2\lambda_2 y \quad (2)$$
$$2z = 2\lambda_1 - \lambda_2 \quad (3)$$
$$x + y + 2z = 2 \quad (4)$$
$$x^2 + y^2 - z = 0 \quad (5)$$

From (1) and (2) we have

$$2x - 2\lambda_2 x = \lambda_1$$
$$2y - 2\lambda_2 y = \lambda_1$$

$$\therefore \quad 2x - 2\lambda_2 x = 2y - 2\lambda_2 y \quad \text{or}$$
$$x - \lambda_2 x = y - \lambda_2 y$$
$$x - \lambda_2 x - y + \lambda_2 y = 0$$
$$x - y - \lambda_2(x - y) = 0$$
$$x - y = \lambda_2(x - y) \quad (6)$$

There are therefore 2 cases to explore:
$x = y$ or $x \neq y$.

Case 1: $(x = y)$

This simplifies (4) as follows:

$2x + 2z = 2 \Rightarrow z = 1 - x$

$2x^2 - z = 0 \Rightarrow z = 2x^2$

$\therefore\ 2x^2 = 1 - x$

$2x^2 + x - 1 = 0$; Solving by quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 8(-1)}}{4} = \frac{-1 \pm 3}{4}$$

$\therefore\ x = \frac{1}{2},\ x = -1$

$y = \frac{1}{2},\ y = -1$

and $z = \frac{1}{2},\ z = 2$

The candidate points are therefore $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$.

$f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$

$f(-1, -1, 2) = 6$

The point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the point nearest to the origin and $(-1, -1, 2)$ is the point farthest from the origin.

Case 2: $x \neq y$

From (6) $x \neq y$ implies that $\lambda_2 = 1$ which in turn means that $\lambda_1 = 0$.

Then, if $\lambda_2 = 1$, $\lambda_1 = 0$, Eq. 3 becomes

$$2z = -1 \Rightarrow z = -\frac{1}{2}$$

However, from Eq. (5) we obtain

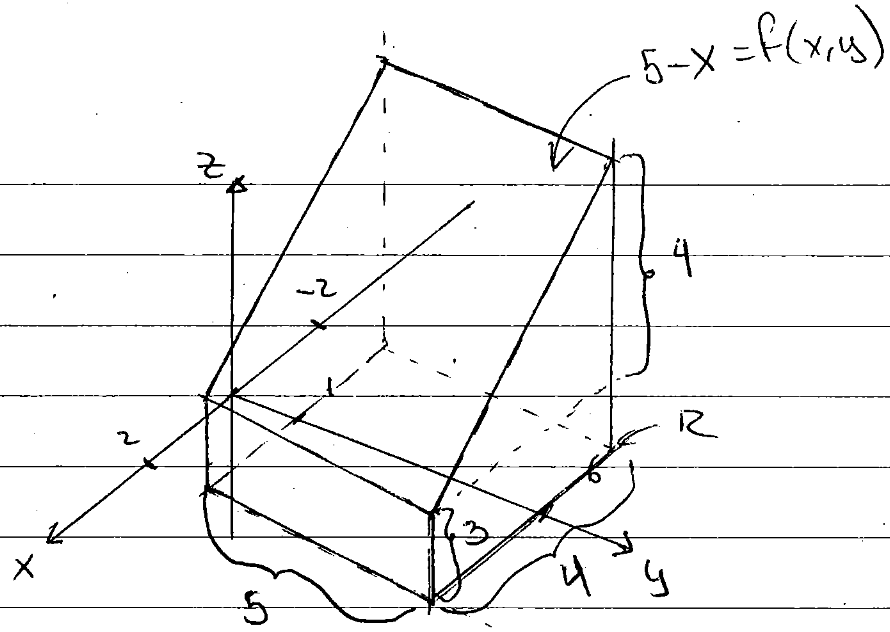
$$x^2 + y^2 = -\frac{1}{2}$$

which is impossible because $x^2 + y^2 \geqslant 0$

This means that x cannot be different than y.

(5) To evaluate $\iint_R (5-x)\,dA$ by interpreting as the volume of a solid we need to see the solid generated by $5-x$ over R in order to determine its dimensions. The solid in question is shown in the following figure.

(5)



The volume of the solid can be calcolated as the sum of the volumes of the base with dimensions $(5 \times 3 \times 4)$ and the triangular prism that is on top of the base. The volume of the prism is equal to the area of the triangular face $\left(\frac{1}{2}(4 \times 4) = 8\right)$ times the length of the prism (5). Therefore, the total volume of the solid is

$$5 \times 3 \times 4 + 8 \times 5 = 60 + 40 = 100 \text{ units}^3$$

(6) $\displaystyle\int_0^2\int_0^{\pi/2} x\sin(y)\,dy\,dx = \int_0^2 -x\Big[\cos(y)\Big]_0^{\pi/2} dx = \int_0^2 -x(0-1)\,dx =$

$$\int_0^2 x\,dx = \frac{x^2}{2}\Big]_0^2 = 2$$

(7) $\int_0^1\int_0^3 e^{x+3y}\,dx\,dy = \int_0^1 \left[e^{x+3y}\right]_0^3 dy = \int_0^1 \left[e^{3+3y} - e^{3y}\right] dy =$

$= \int_0^1 e^{3+3y}\,dy - \int_0^1 e^{3y}\,dy \Rightarrow$ [u-substitution]
$u = 3+3y$
$du = 3dy$
and
$u = 3y$
$du = 3dy$

$= \frac{1}{3}\left[e^{3+3y}\right]_0^1 - \frac{1}{3}\left[e^{3y}\right]_0^1$

$= \frac{1}{3}\left[e^6 - e^3\right] - \frac{1}{3}\left[e^3 - 1\right]$

$= \frac{1}{3}e^6 - \frac{2}{3}e^3 + \frac{1}{3} = \underline{\frac{1}{3}\left(e^3 - 1\right)^2}$

(8) $\iint_D x^3\,dA$, $D = \{(x,y) \mid 1 \le x \le e,\ 0 \le y \le \ln(x)\}$

$\iint_D x^3\,dA = \int_1^e \int_0^{\ln(x)} x^3\,dy\,dx = \int_1^e \left[x^3 y\right]_0^{\ln(x)} dx =$

$\int_1^e \left[x^3 \ln(x) - 0\right] dx = \int_1^e x^3 \ln(x)\,dx \Rightarrow$ [By parts]
$u = \ln(x)$
$v' = x^3\,dx$

$$\int_1^e x^3 \ln(x)\,dx = \ln(x)\frac{x^4}{4}\Big]_1^e - \int_1^e \frac{x^3}{4}\,dx =$$

$$\ln(x)\frac{x^4}{4} - \frac{1}{4}\left(\frac{x^4}{4}\right)\Big]_1^e = \frac{x^4}{4}\left(\ln(x) - \frac{1}{4}\right)\Big]_1^e =$$

$$\frac{e^4}{4}\left(\ln(e) - \frac{1}{4}\right) - \frac{1}{4}\left(\ln(1) - \frac{1}{4}\right) = \frac{3e^4}{16} + \frac{1}{16}$$

9 $\iint_D (x+y)\,dA$ D bounded by $y=\sqrt{x}$ and $y=x^2$

First, we find intersection points:

$$\sqrt{x} = x^2$$
$$x = x^4$$
$$x^4 - x = 0$$
$$x(x^3-1) = 0 \Rightarrow x=0,\ x=1$$

$$\iint_D (x+y)\,dA = \int_0^1\int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx =$$

$$\int_0^1 \left[\left(x^{3/2} + \frac{x}{2}\right) - \left(x^3 + \frac{x^4}{2}\right)\right] dx =$$

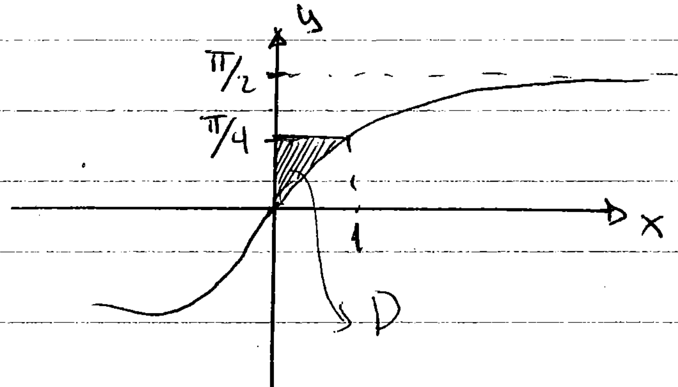
$$\int_0^1 x^{3/2}\,dx + \frac{1}{2}\int_0^1 x\,dx - \int_0^1 x^3\,dx - \frac{1}{2}\int_0^1 x^4\,dx =$$

$$\left[\frac{2}{5}x^{5/2} + \frac{1}{2}\left(\frac{x^2}{2}\right) - \frac{x^4}{4} - \frac{1}{2}\left(\frac{x^5}{5}\right)\right]_0^1 =$$

$$\boxed{\frac{2}{5} + \frac{1}{4} - \frac{1}{4} - \frac{1}{10} = \frac{4-1}{10} = \frac{3}{10}}$$

10.

Sketch:



Changing order of integration:

$$\int_0^1 \int_{\arctan(x)}^{\pi/4} f(x,y)\,dy\,dx = \int_0^{\pi/4}\int_0^{\tan(y)} f(x,y)\,dx\,dy$$