# Homework #13



1. The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region $D$ are given by

$$\bar{x} = \frac{1}{m} \iint_D x\rho(x,y)\, dA \quad \text{and} \quad \bar{y} = \frac{1}{m} \iint_D y\rho(x,y)\, dA$$

where $\rho(x,y)$ is the lamina's density function and the mass $m$ is given by

$$m = \iint_D \rho(x,y)\, dA$$

In this problem, we are told that $\rho(x,y)$ is proportional to the distance between the point $(x,y)$ and the x-axis. Therefore

$$\rho(x,y) = ky.$$

This means that the mass of the lamina is given by

$$M = \iint_D k y \, dA$$

Since $D$ occupies the part of the disk $x^2+y^2 \leq 1$ in the first quadrant, we can set up this double integral as the following iterated integrals: (see figure 1)

$$\iint_D k y \, dA = \int_0^1 \int_0^{\sqrt{1-x^2}} k y \, dy \, dx = \int_0^1 k \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx =$$

$$\frac{k}{2}\int_0^1 \left(\sqrt{1-x^2}\right)^2 dx = \frac{k}{2}\int_0^1 (1-x^2)\,dx = \frac{k}{2}\left[x - \frac{x^3}{3}\right]_0^1 =$$

$$\frac{k}{2}\left(1 - \frac{1}{3}\right) = \frac{2k}{6} = \frac{k}{3}$$

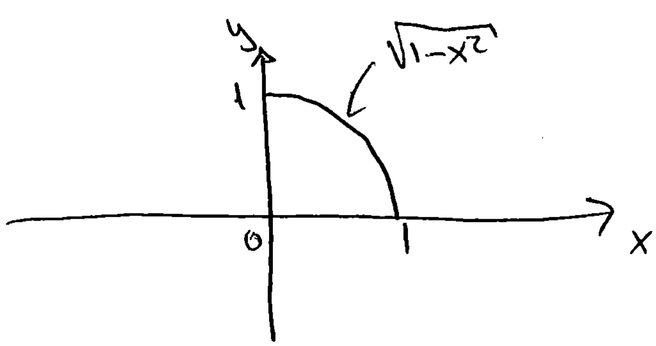



Figure 1.

$$M_y = \iint_D x(Ky)\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} Kxy \; dy\,dx = \int_0^1 Kx\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{K}{2}\int_0^1 x(1-x^2)\,dx = \frac{K}{2}\int_0^1 x\,dx - \frac{K}{2}\int_0^1 x^3\,dx =$$

$$\frac{K}{4}\left[x^2\right]_0^1 - \frac{K}{8}\left[x^4\right]_0^1 = \frac{K}{4} - \frac{K}{8} = \frac{K}{8}$$

$$M_x = \iint_D y(Ky)\,dA = \iint_D Ky^2\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} Ky^2\,dy\,dx$$

This integral will result in

$$\frac{K}{3}\int_0^1 (1-x^2)^{3/2}\,dx$$

which we can simplify if we use polar coordinates.

If we do this, $x = r\cos\theta$, $y = r\sin\theta$ so

$$\int_0^1 \int_0^{\sqrt{1-x^2}} Ky^2\,dy\,dx = \int_0^{\frac{\pi}{2}} \int_0^1 K(r\sin\theta)^2 \, r \, dr\,d\theta$$

Jacobian that results from this transformation

So

$$M_{x} = \int_0^{\frac{\pi}{2}} \int_0^1 K r^3 \sin^2\theta \, dr \, d\theta = \int_0^{\pi/2} K \sin^2\theta \left[\frac{r^4}{4}\right]_0^1 d\theta =$$

$$\frac{K}{4}\int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta.$$

By parts:

$$\int_a^b \sin^2\theta \, d\theta = \begin{bmatrix} u = \sin\theta & u' = \cos\theta \\ v' = \sin\theta \, d\theta & v = -\cos\theta \end{bmatrix} = -\sin\theta\cos\theta\Big]_a^b + \int_a^b \cos^2\theta \, d\theta$$

But $\cos^2\theta = 1 - \sin^2\theta$, So

$$\int_a^b \cos^2\theta \, d\theta = \int_a^b (1 - \sin^2\theta) \, d\theta = \int_a^b d\theta - \int_a^b \sin^2\theta \, d\theta$$

Therefore

$$\int_a^b \sin^2\theta \, d\theta = -\sin\theta\cos\theta\Big]_a^b + \int_a^b d\theta - \int_a^b \sin^2\theta \, d\theta$$

$$2\int_a^b \sin^2\theta \, d\theta = -\sin\theta\cos\theta\Big]_a^b + \int_a^b d\theta$$

$$\int_a^b \sin^2\theta \, d\theta = \frac{1}{2}\left[\int_a^b d\theta - \sin\theta\cos\theta\Big]_a^b\right]$$

Thus:

$$M_x = \frac{K}{4}\left[\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta - \sin\theta\cos\theta\Big]_0^{\frac{\pi}{2}}\right] = \frac{K}{4}\left[\frac{1}{2}\left(\frac{\pi}{2}\right) - 0\right]$$

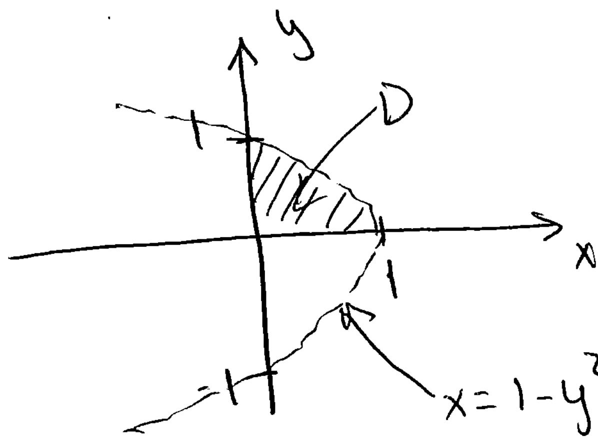
$$= \boxed{\frac{K\pi}{16}}$$

We conclude then that

$$(\bar{x}, \bar{y}) = \left(\frac{\frac{K}{8}}{\frac{K}{3}}, \frac{\frac{K\pi}{16}}{\frac{K}{3}}\right) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$$

2.

$$M = \iint_D \rho(x,y)\, dA$$




$$M = \int_0^1 \int_0^{1-y^2} y \, dx\, dy = \int_0^1 y[x]_0^{1-y^2} dy = \int_0^1 y(1-y^2)\,dy =$$

$$\int_0^1 y\,dy - \int_0^1 y^3\,dy = \frac{y^2}{2} - \frac{y^4}{4}\Big]_0^1 = \frac{1}{2} - \frac{1}{4} = \boxed{\frac{1}{4}}$$

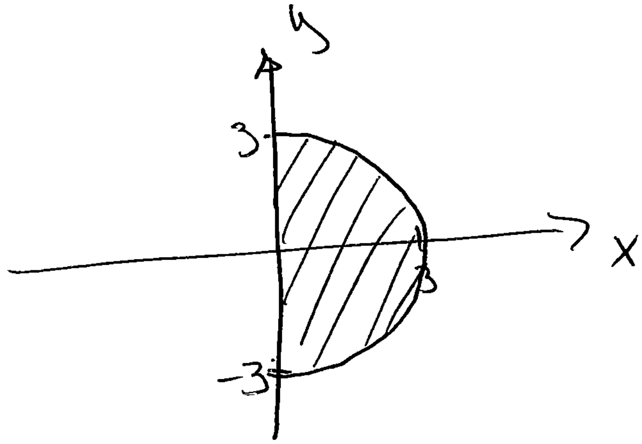
3. Given

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + yx^2)\, dy\, dx$$

we can see that the region of integration is $D = \{(x,y) \mid 0 \leq x \leq 3, -\sqrt{9-x^2}\, y \leq \sqrt{9-x^2}\}$, so

if $-\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2} \Rightarrow |y| \leq \sqrt{9-x^2}$

$\therefore\ y^2 \leq 9 - x^2$ or $x^2 + y^2 \leq 9$. This last expression tells us that the region of integration is half a disk with radius 3.




In polar coordinates, this region of integration is $S = \{(r,\theta) \mid 0 \leq r \leq 3, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$

Now, given the fact that

$x = r\cos\theta$
$y = r\sin\theta$

$$x^3 + y\,x^2 = r^3\cos^3\theta + r\sin\theta\,(r^2\cos^2\theta)$$
$$= r^3\cos^3\theta + r^3\sin\theta\cos^2\theta = r^3\left(\cos^3\theta + \sin\theta\cos^2\theta\right)$$

The integral becomes

$$\underline{I} = \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^3 + yx^2\, dy\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 r^3\left(\cos^3\theta + \sin\theta\cos^2\theta\right) \underset{\text{Jacobian}}{\underset{\downarrow}{r}}\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 r^4\left(\cos^3\theta + \sin\theta\cos^2\theta\right) dr\, d\theta$$

Now

$$\cos^3\theta = \cos^2\theta\cos\theta = \left(1-\sin^2\theta\right)\cos\theta = \cos\theta - \sin^2\theta\cos\theta$$

So

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos\theta - \sin^2\theta\cos\theta + \sin\theta\cos^2\theta\right)\left[\frac{r^5}{5}\right]_0^3 d\theta$$

$$= \frac{243}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos\theta - \sin^2\theta\cos\theta + \sin\theta\cos^2\theta\right) d\theta$$

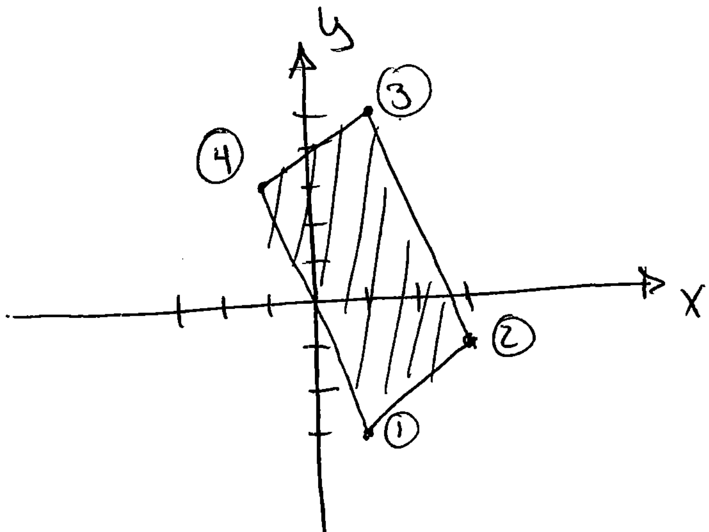
$$= \frac{243}{5}\left[\sin\theta - \frac{\sin^3\theta}{3} - \frac{\cos^3\theta}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$I = \frac{243}{5}\left[\left(1-\frac{1}{3}\right)-\left(-1-\left(-\frac{1}{3}\right)\right)\right]$$

$$= \frac{243}{5}\left[\frac{2}{3}-\left(-1+\frac{1}{3}\right)\right] = \frac{243}{5}\left(\frac{2}{3}-\left(-\frac{2}{3}\right)\right) =$$

$$= \frac{243}{5}\left(\frac{2}{3}+\frac{2}{3}\right) = \frac{243}{5}\left(\frac{4}{3}\right) = \frac{81\cdot 4}{5} = \frac{324}{5}$$

4. The region of integration is the parallelogram with vertices $(1,-3)$, $(3,-1)$, $(1,5)$ and $(-1,3)$




We are given the transformation:

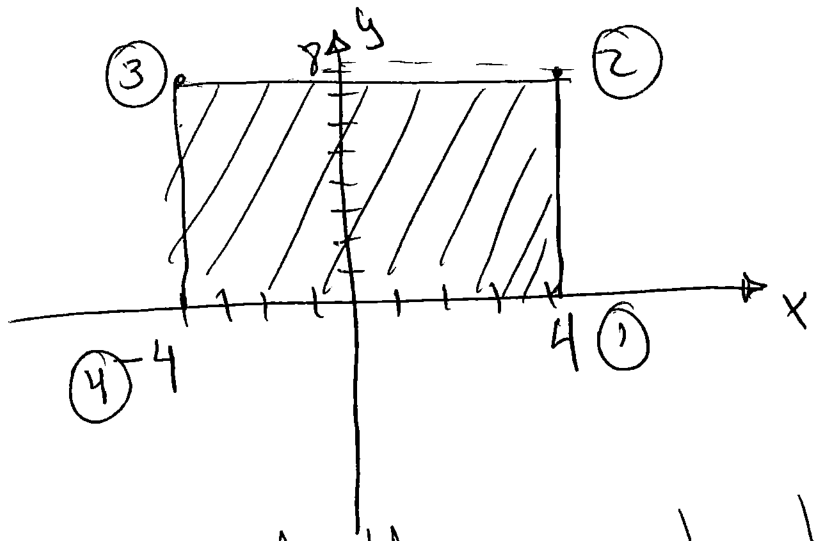
$$x = \frac{1}{4}(u+v) \qquad \text{A}$$

$$y = \frac{1}{4}(v-3u) \qquad \text{B}$$

whose inverse transformation is:

$$\left.\begin{aligned} u &= x-y \\ v &= 3x+y \end{aligned}\right\}$$ You find these by solving for u and v eqs. A and B

Plotting the transformed region:




This means that the new limits of integration are $-4 \le u \le 4$, $0 \le v \le 8$

Now, the function we want to integrate is $4x+8y$, which after the transformation becomes

$$4x+8y = (u+v) + 2(v-3u)$$
$$= u+v+2v-6u$$
$$= -5u+3v$$

The final ingredient to change the original integral, is to compute the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{16} + \frac{3}{16} = \frac{4}{16} = \frac{1}{4}$$

Thus

$$dx\,dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv = \frac{1}{4}\,du\,dv$$

Putting everything together:

$$\iint_R (4x+8y)\,dA = \int_0^8 \int_{-4}^{4} 3v-5u \left(\frac{1}{4}\right) du\,dv$$

$$= \frac{1}{4}\int_0^8 \left[3vu - \frac{5}{2}u^2\right]_{-4}^{4} dv$$

$$= \frac{1}{4}\int_0^8 (12v - 5(8)) - (-12v - 5(8))\,dv$$

$$= \frac{1}{4}\int_0^8 24v\,dv = 6\,\frac{v^2}{2}\Big]_0^8 = 3(64) = \underline{192}$$

5. We are given $\iint_R (x+y)\,e^{x^2-y^2}\,dA$ and

$x-y=0, \quad x-y=2, \quad x+y=0, \quad x+y=3.$

It seems natural to use the following transformation:

$$u = x-y$$
$$v = x+y$$

Then, the new region of integration would be given by the region bounded by

$u=0$, $u=2$

$v=0$, $v=3$.

Additionally, the function $(x+y)e^{x^2-y^2} = (x+y)e^{(x+y)(x-y)}$ would be $ve^{uv}$.

The Jacobian of the transformation is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1=2$$

This means that

$$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy = 2\,dx\,dy$$

and therefore:

$$dx\,dy = \frac{1}{2}\,du\,dv$$

Putting all these elements together:

$$I = \iint_R (x+y)e^{x^2-y^2}\,dA = \int_0^3 \int_0^2 ve^{uv}\frac{1}{2}\,du\,dv =$$

$$I = \frac{1}{2}\int_0^3 \int_0^2 v e^{uv}\, du\, dv \; ; \quad \text{if} \quad \begin{aligned} w &= uv \\ dw &= v\, du \end{aligned}$$

so

$$I = \frac{1}{2}\int_0^3 \int_0^{2v} e^{w}\, dw\, dv = \frac{1}{2}\int_0^3 [e^{2v} - 1]\, dv$$

$$= \frac{1}{2}\int_0^3 e^{2v}\, dv - \frac{1}{2}\int_0^3 dv \; ; \quad \text{if} \quad \begin{aligned} w &= 2v \\ dw &= 2\, dv \end{aligned}$$

so

$$I = \frac{1}{2}\int_0^6 \frac{1}{2} e^{w}\, dw - \frac{3}{2} = \frac{1}{4}[e^6 - 1] - \frac{3}{2}$$

$$= \frac{1}{4}e^6 - \frac{1}{4} - \frac{3}{2}$$

$$= \frac{1}{4}e^6 - \frac{(1+6)}{4}$$

$$= \frac{1}{4}\left(e^6 - 7\right)$$