Homework #14

Typo! (Sorry)

1. Given $\displaystyle\int_{2}^{-2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$

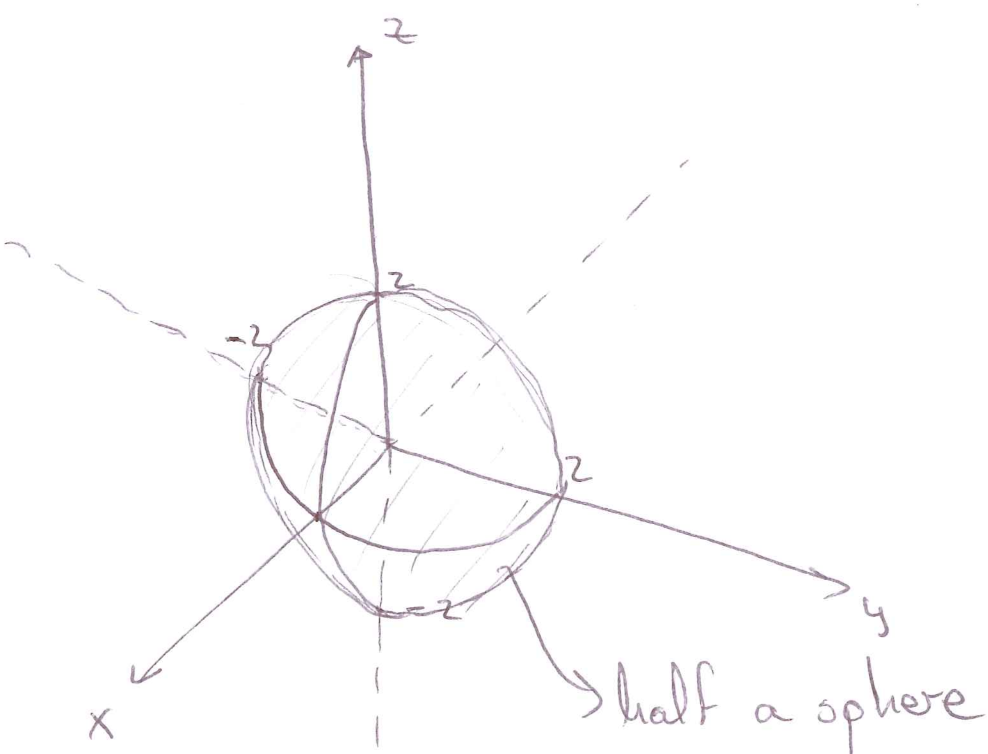
we see that:

$-\sqrt{4-x^2-y^2} \leq z \sqrt{4-x^2-y^2}$ $\qquad z^2+x^2+y^2 \leq 4$

$0 \leq x \leq \sqrt{4-y^2}$ which means $x^2+y^2 \leq 4,\ x \geq 0$

$-2 \leq y \leq 2$ $\qquad -2 \leq y \leq 2$

So the region of integration is



Spherical coordinates transformation:

$z = r\cos\phi$; $x = r\sin\phi\cos\theta$; $y = r\sin\phi\sin\theta$

In polar coordinates, this region is given by $D = \{(r, \phi, \theta) \mid 0 \le r \le 2,\ 0 \le \phi \le \pi,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$

The function $y^2\sqrt{x^2+y^2+z^2}$ becomes

$$(r \sin\phi \sin\theta)^2 \, r = r^3 \sin^2\phi \sin^2\theta$$

The Jacobian of this transformation is:

$$r^2 \sin\phi$$

So, the integral we want to evaluate is

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \int_0^2 r^5 \sin^3\phi \sin^2\theta \, dr \, d\phi \, d\theta =$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \sin^3\phi \sin^2\theta \left[\frac{r^6}{6}\right]_0^2 d\phi \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \frac{32}{3} \sin^3\phi \sin^2\theta \, d\phi \, d\theta =$$

$$\frac{32}{3} \int_{-\frac{\pi}{2}}^{\pi/2} \sin^2\theta \left[\int_0^{\pi} \sin^3\phi \, d\phi\right] d\phi$$

$$\int_0^{\pi} \sin^3\phi \, d\phi = \int_0^{\pi} (1-\cos^2\phi)\sin\phi \, d\phi = \int_0^{\pi} \sin\phi \, d\phi - \int_0^{\pi} \cos^2\phi \sin\phi \, d\phi$$

$$\text{if } u = \cos\phi$$
$$\Rightarrow du = -\sin\phi\, d\phi$$

So

$$\int_0^{\pi} \sin\phi\, d\phi - \int_0^{\pi} \cos^2\phi \sin\phi\, d\phi = -\cos\phi\Big]_0^{\pi} + \int_1^{-1} u^2\, du$$

$$= -(-1) + (1) + \frac{u^3}{3}\Big]_1^{-1}$$

$$= 2 + \left(\frac{-1}{3} - \frac{1}{3}\right) = \frac{6}{3} - \frac{2}{3} = \frac{4}{3}$$

Then

$$I = \frac{32}{3} \int_{-\frac{\pi}{2}}^{\pi/2} \sin^2\theta \left(\frac{4}{3}\right) d\theta = \frac{128}{9} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\, d\theta = \frac{128}{9} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta \sin\theta\, d\theta$$

By parts:

$u = \sin\theta \quad u' = \cos\theta\, d\theta$

$v' = \sin\theta\, d\theta \quad v = -\cos\theta$

$$\Rightarrow \int \sin^2\theta\, d\theta = -\sin\theta\cos\theta + \int \cos^2\theta\, d\theta$$

$$\int \sin^2\theta\, d\theta = -\sin\theta\cos\theta + \int (1 - \sin^2\theta)\, d\theta$$

$$\int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta$$

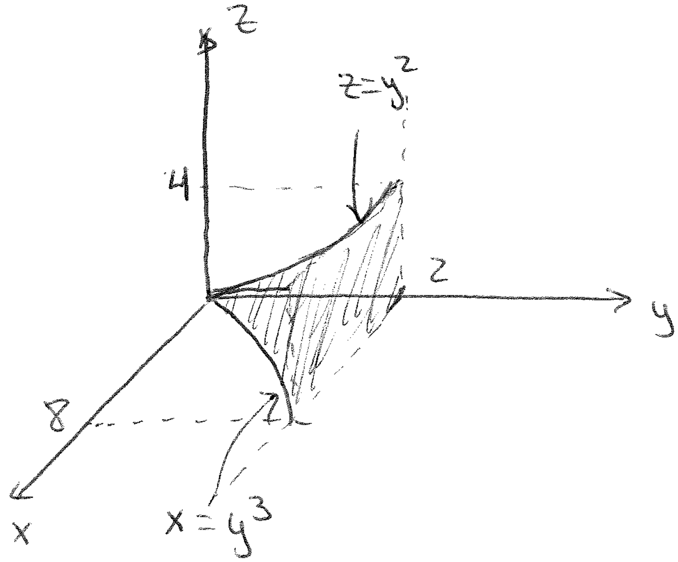
$$\therefore I = \frac{128}{9}\left[\frac{1}{2}\left(\theta - \sin\theta\cos\theta\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{64}{9}\left[\left(\frac{\pi}{2} - 0\right) - \left(-\frac{\pi}{2} - 0\right)\right]$$

$$= \frac{64}{9}\pi$$

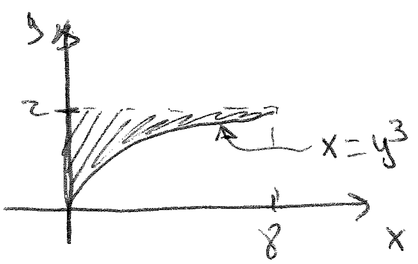
2. We are given

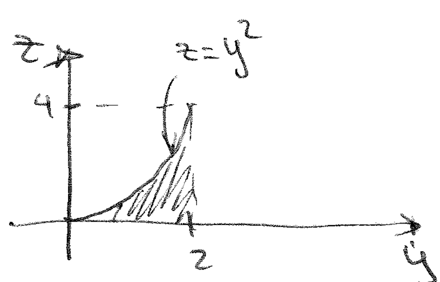
$$I = \int_0^2 \int_0^{y^3} \int_0^{y^2} f(x,y,z)\, dz\, dx\, dy$$

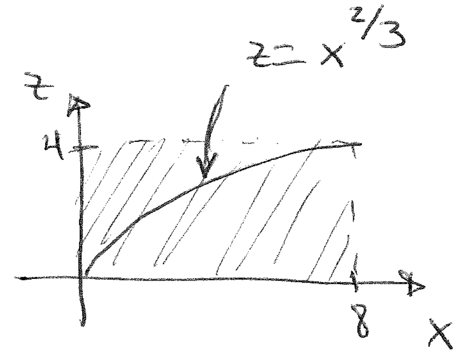
Plotting this region:



Decomposing in plane projections:







This is because

$$0 \le x \le y^3$$
$$0 \le z \le y^2$$

So $\sqrt[3]{x} \le y$

$$\sqrt{z} \le y$$

On the boundary:

$$\sqrt{z} \le \sqrt[3]{x} \Rightarrow z \le x^{2/3}$$

Now we can find different orders:

1.

$$I = \int_0^2 \int_0^{y^2} \int_0^{y^3} f(x,y,z)\, dx\, dz\, dy$$

2.

$$I = \int_0^8 \int_{\sqrt[3]{x}}^{2} \int_0^{y^2} f(x,y,z)\, dz\, dy\, dx$$

3.

$$I = \int_0^4 \int_{\sqrt{z}}^{2} \int_0^{y^3} f(x,y,z)\, dx\, dy\, dz$$

4.

$$I = \int_0^8 \int_0^{x^{2/3}} \int_{\sqrt[3]{x}}^{2} f(x,y,z)\, dy\, dz\, dx + \int_0^8 \int_{x^{2/3}}^{4} \int_{\sqrt{z}}^{2} f(x,y,z)\, dy\, dz\, dx$$

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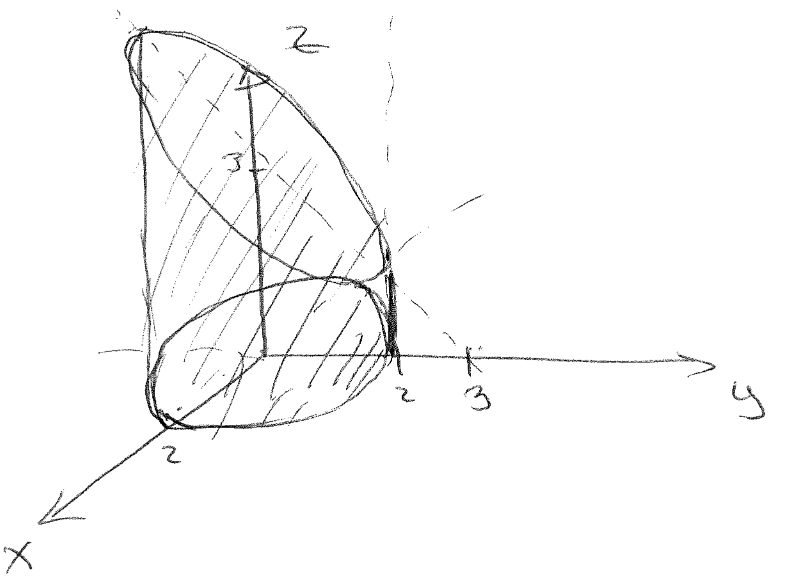
$$I = \int_0^4 \int_0^{z^{3/2}} \int_{\sqrt{z}}^{2} f(x,y,z)\, dy\, dx\, dz + \int_0^4 \int_{z^{3/2}}^{8} \int_{\sqrt[3]{x}}^{2} f(x,y,z)\, dy\, dx\, dz$$

3. The volume we are looking for is sketched below



In cylindrical coordinates, this volume is bounded by

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

So $x^2+y^2=4$

$$r^2\cos^2\theta + r^2\sin^2\theta = 4$$

$$r^2 = 4$$

$$r = 2$$

$$x + z = 3$$

$$z = 3 - x$$

$$= 3 - r\sin\theta$$

So

$$V = \int_0^2 \int_0^{2\pi} \int_0^{3-r\sin\theta} \underbrace{r}_{\text{Jacobian}}\, dz\, d\theta\, dr = \int_0^2 \int_0^{2\pi} r(3 - r\sin\theta)\, d\theta\, dr$$

$$= \int_0^2 \int_0^{2\pi} (3r - r^2\sin\theta)\, d\theta\, dr = \int_0^2 3r\theta\Big]_0^{2\pi} + \cancel{r^2\cos\theta\Big]_0^{2\pi}} \to 0 \; dr$$

$$V = \int_0^2 3r(2\pi)\, dr = 6\pi \int_0^2 r\, dr = 3\pi\, r^2\Big]_0^2 = \underline{12\pi}$$

4. Using the given transformation, the region of integration is bounded by the coordinate planes and $u+v+w=1$.

The Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw$$

and since $u \geq 0$, $v \geq 0$ and $w \geq 0$ then

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = 8uvw.$$

$$V = \iiint_E dV = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw\, dw\, dv\, du$$

$$= \int_0^1 \int_0^{1-u} 8uv\left[\frac{w^2}{2}\right]_0^{1-u-v} dv\, du$$

$$= 4\int_0^1 \int_0^{1-u} uv(1-u-v)^2\, dv\, du$$

Now

$$(1-u-v)^2 = ((1-u)-v)^2 = (1-u)^2 - 2(1-u)v + v^2$$

So $uv(1-u-v)^2 = u(1-u)^2 v - 2u(1-u)v^2 + uv^3$

$$V = \int_0^1 4 \int_0^{1-u} \left[u(1-u)^2 v - 2u(1-u)v^2 + uv^3\right] dv\, du$$

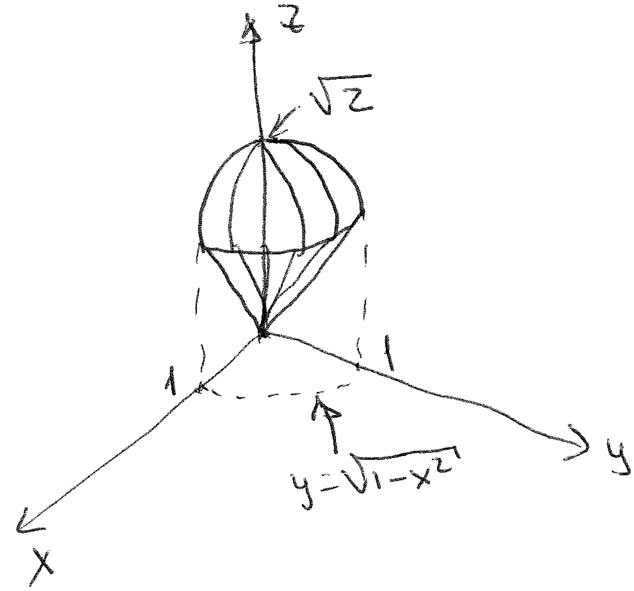
$$= \int_0^1 \left[2u(1-u)^4 - \frac{8}{3}u(1-u)^4 + u(1-u)^4\right] du$$

$$= \int_0^1 \frac{1}{3} u(1-u)^4\, du = \int_0^1 \frac{1}{3}\left[(1-u)^4 - (1-u)^5\right] du$$

$$= \frac{1}{3}\left[-\frac{1}{5}(1-u)^5 + \frac{1}{6}(1-u)^6\right]_0^1 = \frac{1}{3}\left(-\frac{1}{6} + \frac{1}{5}\right) = \underline{\frac{1}{90}}$$

5. The region of integration is bounded in the z direction by the cone $z = \sqrt{x^2+y^2}$ from below and by the sphere $x^2+y^2+z^2 = 2$ from above. The other limits of integration are telling us that x and y are positive. Therefore, the region of integration is in the

first octant and looks like a quarter of an ice cream.



Since the volume is in the first octant

$$0 \le \theta \le \frac{\pi}{2}$$

The radius of the sphere is $\sqrt{2}$, so

$$0 \le r \le \sqrt{2}$$

Finally; the equation of the cone is

$$z^2 = x^2 + y^2$$
$$r^2\cos^2\phi = r^2\sin^2\phi \Rightarrow \tan\phi = 1 \Rightarrow \phi = \frac{\pi}{4}$$

so $\quad 0 \le \phi \le \frac{\pi}{4}$

The integral in spherical coordinates becomes

$$V = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} (r\sin\phi\cos\theta)(r\sin\phi\sin\theta)\underbrace{r^2\sin\phi}_{\text{Jacobian}}\,dr\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{4}} \sin^3\phi \left[\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \left[\int_0^{\sqrt{2}} r^4\,dr\right] d\theta\right] d\phi$$

$$= \frac{4\sqrt{2}}{5}\int_0^{\frac{\pi}{4}} \sin^3\phi \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta\,d\phi$$

$$= \frac{4\sqrt{2}}{5}\int_0^{\frac{\pi}{4}} \sin^3\phi \left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} d\phi = \frac{4\sqrt{2}}{5}\int_0^{\frac{\pi}{4}} \sin^3\phi\left(\frac{1}{2}\right)d\phi$$

$$= \frac{2\sqrt{2}}{5}\int_0^{\frac{\pi}{4}} \sin^3\phi\,d\phi = \frac{2\sqrt{2}}{5}\left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\frac{\pi}{4}}$$

$$= \frac{2\sqrt{2}}{5}\left[\frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{2} - \left(\frac{1}{3} - 1\right)\right] = \frac{4\sqrt{2} - 5}{15}$$

6. A sphere of radius a in spherical coordinates is given by $r = a$. Therefore the given integral is equal to

$$I = \lim_{a\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r e^{-r^2}\underbrace{r^2\sin\phi}_{\text{Jacobian}}\,dr\,d\phi\,d\theta$$

$$I = \lim_{a\to\infty} \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^3 e^{-r^2} \sin\phi \, dr \, d\phi \, d\theta$$

Let's focus on

$$\int_0^a r^3 e^{-r^2} dr \qquad u' = e^{-r^2} r \qquad u = -\frac{1}{2}e^{-r^2}$$

$$v = r^2 \qquad v' = 2r$$

So

$$\int_0^a r^3 e^{-r^2} dr = -\frac{r^2}{2} e^{-r^2}\Big]_0^a + \int_0^a r e^{-r} = -\frac{r^2}{2} e^{-r^2} - \frac{1}{2} e^{-r^2}\Big]_0^a$$

$$= \left(\frac{-a^2}{2} e^{-a^2} - \frac{1}{2} e^{-a^2}\right) - \left(0 - \frac{1}{2} e^0\right)$$

$$= \frac{1}{2}\left(-a^2 e^{-a^2} - e^{-a^2} + 1\right)$$

So

$$I = \lim_{a\to\infty} \int_0^{2\pi} \frac{1}{2} \int_0^{\pi} (-a^2 e^{-a^2} - e^{-a^2} + 1) \sin\phi \, d\phi \, d\theta$$

$$= \lim_{a\to\infty} \int_0^{2\pi} \frac{1}{2}\left(-a^2 e^{-a^2} - e^{-a^2} + 1\right) \int_0^{\pi} \sin\theta \, d\phi \, d\theta$$

$$= \lim_{a\to\infty} \int_0^{2\pi} \frac{1}{2}\left(-a^2 e^{-a^2} - e^{-a^2} + 1\right)(2) \, d\theta$$

$$= \lim_{a\to\infty} 2\pi\left(-a^2 e^{-a^2} - e^{-a^2} + 1\right) = \boxed{2\pi}$$

[By L'hopital's rule $\lim_{a\to\infty} a^2 e^{-a^2} = 0$]

7.- In this case, it is convenient to switch from Cartesian to cylindrical coordinates. If we do so, the region of integration is bounded by the cylinder with equation $r^2\cos^2\theta + r^2\sin^2\theta = 1 = r^2 \Rightarrow r = 1$ and from above and below by the sphere $\underbrace{r^2}_{} + z^2 = 4$

we just saw that $x^2+y^2=r^2$

so the volume is given by:

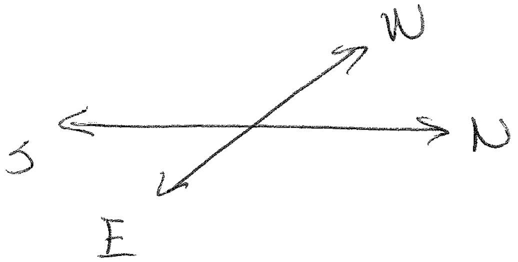
$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

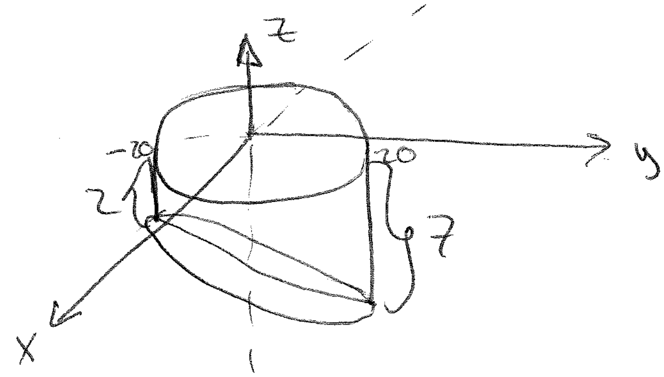
$$= \int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[-\frac{2}{3}(4-r^2)^{3/2}\right]_0^1 d\theta$$

$$= 2\pi\left[-\frac{2}{3}(4-r^2)^{3/2}\right]_0^1 = \frac{4}{3}\pi\left(8-3^{3/2}\right)$$

8.





Slope of pool: $-\frac{7-2}{40} = -\frac{5}{40} = -\frac{1}{8}$, therefore the plane that represents the bottom of the pool is $z = -\frac{1}{8}y \underbrace{- \frac{9}{2}}_{\substack{\text{depth} \\ @ y=0}}$

In cylindrical coordinates, this plane is

$$z = -\frac{1}{8} r\sin\theta - \frac{9}{2}$$

The volume of the pool is:

$$V = \int_0^{2\pi} \int_0^{20} \int_{-\frac{1}{8}r\sin\theta - \frac{9}{2}}^{0} r \, dz \, dr \, d\theta$$

$$V=\int_0^{2\pi}\int_0^{20} r\left(\frac{1}{8}r\sin\theta+\frac{9}{2}\right)dr\,d\theta$$

$$=\int_0^{2\pi}\left[\frac{r^3}{24}\sin\theta+\frac{9}{4}r^2\right]_0^{20}d\theta$$

$$=\int_0^{2\pi}\left[\frac{1000}{3}\sin\theta+900\right]d\theta=\cancel{\frac{1000}{3}\left(-\cos\theta\right)\Big]_0^{2\pi}}^{\,0}+900(2\pi)$$

$$\boxed{=1800\pi}$$

9. The absolute value of the Jacobian plays the role of a scaling factor between areas in the original space x, y, z and the transformed space u, v, w. The relationships are

$$dx\,dy\,dz=\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|du\,dv\,dw$$

$$du\,dv\,dw=\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|dx\,dy\,dz$$

which means that:

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|=\frac{1}{\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|}$$

10.- Fubini's theorem will hold as long as $f(x,y)$ is bounded on the region of integration, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

In class we saw an example where one of the iterated integrals does not converge. This is the result of $f(x,y)$ not being bounded.

Another example is $f(x,y) = \frac{\sin(x)}{y}$ on the region $[0,1] \times [0,2\pi]$.

$$\int_0^1 \int_0^{2\pi} \frac{\sin(x)}{y}\, dx\, dy = \int_0^1 \frac{1}{y}\left[\int_0^{2\pi} \sin(x)\, dx\right] dy$$

$$= \int_0^1 0\, dy = 0$$

$$\int_0^{2\pi} \int_0^1 \frac{\sin(x)}{y}\, dy\, dx = \int_0^{2\pi} \sin(x) \underbrace{\Big[\ln(y)\Big]_0^1}_{\substack{\text{Doesn't exist} \\ \text{because } \ln(0) \\ \text{doesn't exist.}}} dx$$