Homework #15

1. At $t=3$, the particle's velocity is $\vec{v}(2,1) = \langle 4, 3 \rangle$. After 0.01 seconds, the particle's displacement is approximately $0.01\,\vec{v}(2,1) = 0.01\langle 4, 3\rangle = \langle 0.04, 0.03 \rangle$. Therefore, the particle's position at $t = 3.01$ will be approximately $\langle 2, 1\rangle + \langle 0.04, 0.03\rangle$

$$\boxed{= \langle 2.04, 1.03 \rangle}$$

2. If $x = 2\sin(t)$, $y = t$, and $z = -2\cos t$, then $xyz = -4t\sin t\cos t$.

Recall that $\frac{ds}{dt} = |\vec{r}\,'(t)|$, that is, speed = magnitude of velocity. Then

$ds = |\vec{r}\,'(t)|\,dt$

Therefore, computing $|\vec{r}\,'(t)|$ we get:

$\vec{r}(t) = \langle 2\sin t, t, -2\cos t \rangle$

$\vec{r}\,'(t) = \langle 2\cos t, 1, 2\sin t \rangle$

$$|\vec{r}'(t)| = \sqrt{4\cos^2 t + 1 + 4\sin^2 t}$$

$$= \sqrt{4+1} = \sqrt{5}$$

Thus

$$I = \int_C xyz\, ds = \int_0^{\pi} -4t\sin t\cos t\,(\sqrt{5})\, dt$$

Recalling that $\sin(2t) = 2\sin t\cos t$

$$I = -2\sqrt{5}\int_0^{\pi} t\sin(2t)\, dt$$

By parts: $u = t$ $\quad u' = 1$
$v' = \sin 2t\, dt$ $\quad v = -\cos 2t\left(\frac{1}{2}\right)$

$$I = -2\sqrt{5}\left[-\frac{t}{2}\cos(2t) + \frac{1}{4}\sin 2t\right]_0^{\pi}$$

$$= -2\sqrt{5}\left(-\frac{\pi}{2} - 0\right) = \underline{\sqrt{5}\,\pi}$$

3. First of all, $\vec{F}$ is defined everywhere in $\mathbb{R}^2$.
Now, since

$$\frac{\partial}{\partial y}\left[1 - ye^{-x}\right] = -e^{-x} = \frac{\partial}{\partial x}\left[e^{-x}\right] = -e^{-x}$$

$\vec{F} = \nabla f$, and thus it is conservative

If $1 - ye^{-x} = \frac{\partial f}{\partial x}$ for some f, then a candidate function is

[Partial integration wrt. x] $x + ye^{-x} + C(y)$

$$\downarrow \frac{\partial}{\partial y}$$

$C_y(y) + e^{-x}$, which is equal to the given component, except for the fact that $C_y(y) = 0 \Rightarrow C(y) = K$

So $f(x,y) = x + ye^{-x} + K$

Now

$$\int_C \vec{F} \cdot d\vec{r} = f(1,2) - f(0,1) = (1 + 2e^{-1}) - (0 + e^{-0})$$

$$= 2e^{-1} = \frac{2}{e}$$

4. The trajectory describes the following path: $(0,0,0) \xrightarrow[C_1]{} (0,0,1) \xrightarrow[C_2]{} (0,1,1) \xrightarrow[C_3]{} (2,1,1)$

If we decompose C into C_1, C_2, C_3 we have

$$I = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$= \int_{C_1} P\,dx + Q\,dy + R\,dz + \int_{C_2} P\,dx + Q\,dy + R\,dz$$

$$+ \int_{C_3} P\,dx + Q\,dy + R\,dz$$

Since y and x remain unchanged along the straight line C_1, $dy = dz = 0$. A similar argument is used for C_2 and C_3.

$$I = \int_{C_1} R\,dz + \int_{C_2} Q\,dy + \int_{C_3} P\,dx$$

$$= \underbrace{\int_0^1 (yz - x)\,dz}_{\substack{\text{along } C_1 \\ y = x = 0}} + \underbrace{\int_0^1 xz\,dy}_{\substack{\text{along } C_2 \\ x = 0,\ z = 1}} + \underbrace{\int_0^2 (2y+3)\,dx}_{\substack{\text{along } C_3 \\ y = 1}}$$

$$= 0 + 0 + \int_0^2 5\,dx = 5x\Big]_0^2 = \underline{10}$$

5. $\vec{r}(t) = \langle 2t^2, t, 4t^2 - t \rangle$

$\vec{r}\,'(t) = \langle 4t, 1, 8t - 1 \rangle$

$\vec{F}(\vec{r}(t)) = \langle 3(2t^2)^2, 2(2t^2)(4t^2 - t) - t, 4t^2 - t \rangle$

$= \langle 12t^4, 16t^4 - 4t^3 - t, 4t^2 - t \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 48t^5 + 16t^4 - 4t^3 - t + 32t^3 - 12t^2 + t$

$= 48t^5 + 16t^4 + 28t^3 - 12t^2$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (48t^5 + 16t^4 + 28t^3 - 12t^2)\,dt$

$= 8t^6 + \frac{16}{5}t^5 + 7t^4 - 4t^3 \Big]_0^1$

$= 8 + \frac{16}{5} + 7 - 4$

$= 11 + \frac{16}{5} = 14.2$

6. $\vec{F} = -185\hat{k}$. The staircase can be parametrized as follows:

$$\vec{r}(t) = \langle 20\cos t, 20\sin t, \frac{90}{6\pi}t \rangle$$

$$0 \leq t \leq 6\pi$$

$$\vec{r}\,'(t) = \langle -20\sin t, 20\cos t, \frac{90}{6\pi} \rangle$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle 0,0,-185 \rangle \cdot \langle -20\sin t, 20\cos t, \frac{90}{6\pi} \rangle$$

$$= -185\left(\frac{90}{6\pi}\right)$$

$$\therefore W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{6\pi} -185\left(\frac{90}{6\pi}\right) dt = -185(90) = \boxed{-16650 \text{ ft-lb}}$$

7. Now $\vec{F} = -\left(185 - \frac{9}{6\pi}t\right)\hat{k} = \left(-185 + \frac{3}{2\pi}t\right)\hat{k}$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle 0, 0, -185 + \frac{3}{2\pi}t \rangle \cdot \langle -20\sin t, 20\cos t, \frac{90}{6\pi} \rangle$$

$$= -185\frac{90}{6\pi} + \frac{90}{4\pi^2}t$$

$$W = \int_C \vec{F} \cdot d\vec{r} = -185\frac{90}{6\pi}\int_0^{6\pi} dt + \frac{90}{4\pi^2}\int_0^{6\pi} t\, dt$$

$$= -185\frac{90}{6\pi}(6\pi) + \frac{90}{4\pi^2}\left(\frac{t^2}{2}\right)_0^{6\pi}$$

$$= -185(90) + \frac{90}{4\pi^2}\left(\frac{36\pi^2}{2}\right) = -185(90) + (45)9 = \boxed{-16245 \text{ ft-lb}}$$

8. $\vec{F} = \langle P, Q, R \rangle = \langle y^2\cos(x) + z^3,\ 2y\sin(x) - 4,\ 3xz^2 + 2 \rangle$

If $\vec{F} = \nabla f$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} : \ 2y\cos(x) = 2y\cos(x)$$

$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} : \ 3z^2 = 3z^2$$

$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} : \ 0 = 0$$

Since $\vec{F}$ is defined everywhere in $\mathbb{R}^2$, we can use the fundamental theorem of calculus for line integrals.

Let's find f:

$$\int (y^2\cos(x) + z^3)\,dx = y^2\sin(x) + xz^3 + C(y,z)$$

Differentiating wrt y:

$2y\sin(x) + C_y(y,z)$, which means $C_y(y,z) = -4$

$\Rightarrow C(y,z) = -4y + W(z)$

Differentiating wrt z:

$3xz^2 + W'(z) = 3xz^2 + 2 \Rightarrow W'(z) = 2 \Rightarrow W(z) = 2z + K$

$f(x,y,z) = y^2\sin(x) + xz^3 - 4y + 2z + K$

$$\int_C \vec{F}\cdot d\vec{r} = f\left(\frac{\pi}{2}, -1, 2\right) - f(0, 1, -1)$$

$$= (-1)^2 \sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2}(2)^3 - 4(-1) + 2(2)$$

$$- \left[(1)^2 \sin(0) + (0)(-1)^3 - 4(1) + 2(-1)\right]$$

$$= 1 + 4\pi + 4 + 4 - (0 + 0 - 4 - 2)$$

$$= 4\pi + 9 + 6 = \boxed{4\pi + 15}$$

9. $$\int_{C_1} \nabla f \cdot d\vec{r} = f(P_f) - f(P_i) = 0$$

$$\Rightarrow f(P_f) = f(P_i)$$

$$\sin(x_f - 2y_f) = \sin(x_i - 2y_i)$$

So one possibility:

$$x_f = 2y_f = x_i - 2y_i$$

$$x_f - x_i = 2y_f - 2y_i \Rightarrow \frac{x_f - x_i}{y_f - y_i} = 2$$

This means that one curve that satisfies $\int_{C_1} \nabla f \cdot d\vec{r} = 0$ is a line with slope $= \frac{1}{2}$

For example:

$$\vec{r}(t) = \langle 2t, t \rangle \qquad 0 \le t \le 1$$

When $\int_{C_2} \nabla f \cdot d\vec{r} = 1$

we have:

$$f(P_f) - f(P_i) = 1$$

$$\sin(x_f - 2y_f) - \sin(x_i - 2y_i) = 1$$

If $x_i = y_i = 0$, and $x_f = \frac{\pi}{2}$ and $y_f = 0$

then $\int_{C_2} \nabla f \cdot d\vec{r} = 1$.

So, the curve C_2 can be parametrized as follows:

$$\vec{r}(t) = \langle \tfrac{\pi}{2} t, 0 \rangle \qquad 0 \le t \le 1$$

10. $I = \oint_C (x-y)\,dx + (x+y)\,dy$

C: $x^2 + y^2 = 4$

Changing to polar coordinates: $x = 2\cos\theta$; $dx = -2\sin\theta\,d\theta$
$y = 2\sin\theta$; $dy = 2\cos\theta\,d\theta$

$$I = \int_0^{2\pi} (2\cos\theta - 2\sin\theta)(-2\sin\theta\,d\theta) + (2\cos\theta + 2\sin\theta)(2\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} (\cancel{-4\sin\theta\cos\theta} + 4\sin^2\theta + 4\cos^2\theta + \cancel{4\sin\theta\cos\theta})\,d\theta = 4\int_0^{2\pi} d\theta = \boxed{8\pi}$$