Homework #3

1. $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k}$
   $\vec{b} = 4\hat{i} + 5\hat{k}$

$$\vec{a} \cdot \vec{b} = (3)(4) + (2)(0) + (-1)(5)$$
$$= 12 + 0 - 5 = \boxed{7}$$

2. Let $\vec{u} = \hat{i}$, $\vec{v} = \frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}$, and $\vec{w} = -\hat{j}$

Thus, $\vec{u} \cdot \vec{v} = (1)\left(\frac{1}{2}\right) + (0)\left(-\frac{1}{2}\right) = \boxed{\frac{1}{2}}$

$\vec{u} \cdot \vec{w} = (1)(0) + (0)(-1) = \boxed{0}$

3. $P(1,-3,-2)$, $Q(2,0,-4)$, $R(6,-2,-5)$

Let's define the vectors $\vec{PQ}$, $\vec{PR}$, and $\vec{QR}$.

$$\vec{PQ} = \langle 2-1, 0-(-3), -4-(-2) \rangle = \langle 1, 3, -2 \rangle$$
$$\vec{PR} = \langle 6-1, -2-(-3), -5-(-2) \rangle = \langle 5, 1, -3 \rangle$$
$$\vec{QR} = \langle 6-2, -2-0, -5-(-4) \rangle = \langle 4, -2, -5 \rangle$$

By trying different combinations of vectors we see that

$$\overrightarrow{PQ} \cdot \overrightarrow{QR} = (1)(4) + (3)(-2) + (-2)(-1) =$$
$$4 - 6 + 2 = 0$$

$\therefore$ The sides PQ and QR of the triangle PQR are perpendicular to each other. This means that PQR is a right triangle.

4. First, we need to find the intersection points. We do this by equating

$$x^2 = x^3$$
$$x^3 - x^2 = 0$$
$$x^2(x-1) = 0$$

Thus, the solutions of this equation are:

$x_1 = 0$

$x_2 = 1$

The intersection points are:

$$\underbrace{P_1(0,0)}_{} \quad \text{and} \quad \underbrace{P_2(1,1)}_{}$$

$x^2|_0 = 0^2 = 0$ $\qquad$ $x^2|_1 = 1^2 = 1$

$x^3|_0 = 0^3 = 0$ $\qquad$ $x^3|_1 = 1^3 = 1$

At P_1, both tangent lines are horizontal, thus the angle between $y=x^2$ and $y=x^3$ at P_1 is zero.

At P_2, the tangent lines are:

For $y=x^2$: The slope of the tangent line is $y'=2x$, which at P_2 is $2(1)=2$. Thus the equation of the tangent line to $y=x^2$ at P_2 is

$$y-1=2(x-1)=2x-2$$
$$y=2x-2+1=2x-1 \quad (1)$$

For $y=x^3$: The slope of the tangent line is $y'=3x^2$, which at P_2 is $3(1)^2=3$. The equation of the tangent line to $y=x^3$ at P_2 is

$$y-1=3(x-1)=3x-3$$
$$y=3x-3+1=3x-2 \quad (2)$$

Let us now find two vectors parallel to the tangent lines. This is done by finding to point on the tangent lines.

For $y=x^2$, the tangent line is $y=2x-1$, thus if we set $x=2$ then $y=2(2)-1=4-1=3$. The point $(2,3)$ is on the tangent line to $y=x^2$. A vector parallel to $y=2x-1$ is therefore

$$\vec{v}_1 = \langle 2-1, 3-1 \rangle = \langle 1, 2 \rangle$$

~~from~~ P_2

For $y=x^3$ we do something similar. If $x=2$, $y=3(2)-2=6-2=4$. Thus $(2,4)$ is on the line tangent to $y=x^3$. A vector parallel to $3x-2=y$ is

$$\vec{v}_2 = \langle 2-1, 4-1 \rangle = \langle 1, 3 \rangle$$

The angle between the curves is equal to the angle between the tangent lines, which is equal to the angle between $\vec{v}_1$ and $\vec{v}_2$.

Using the fact that

$$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1||\vec{v}_2| \cos\theta$$

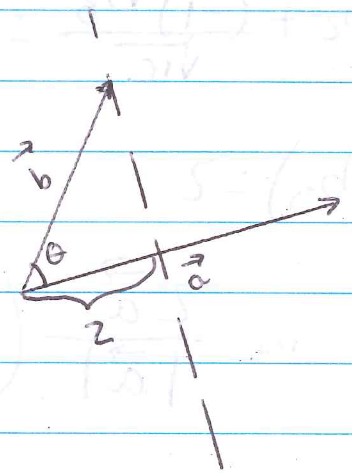
where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$ we have

$$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \frac{1+6}{\sqrt{1^2+2^2}\sqrt{1^2+9^2}} = \frac{7}{\sqrt{5}\sqrt{10}} =$$

$$\frac{7}{\sqrt{50}} = \frac{7}{5\sqrt{2}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right)}$$

5.



We are looking for a vector whose tip touches the dotted line in the figure above. Note that there are many such vectors, but they all satisfy

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{\vec{a}}{|\vec{a}|}\cdot\vec{b} = 2$$

Let's find $\frac{\vec{a}}{|\vec{a}|}$. This vector is

$$\frac{\langle 3, 0, -1\rangle}{\sqrt{3^2+0^2+(-1)^2}} = \frac{\langle 3, 0, -1\rangle}{\sqrt{9+0+1}} = \frac{\langle 3, 0, -1\rangle}{\sqrt{10}}$$

Therefore, the vector $\vec{b} = \langle b_1, b_2, b_3\rangle$ that we are looking for must satisfy

$$\frac{3b_1}{\sqrt{10}} + 0b_2 + \frac{(-1)b_3}{\sqrt{10}} = 2 \quad \text{or}$$

$$\frac{1}{\sqrt{10}}(3b_1 - b_3) = 2$$

One such vector is $\frac{2\vec{a}}{|\vec{a}|}$ (basically $\text{proj}_{\vec{a}}\vec{b}$)

or

$$\vec{b} = \left\langle \frac{6}{\sqrt{10}}, 0, -\frac{2}{\sqrt{10}} \right\rangle$$

6. Work, denoted by w, is defined as $w = \vec{F} \cdot \vec{D}$, where $\vec{F}$ is the force vector acting upon an object, and $\vec{D}$ is the displacement vector that is produced by $\vec{F}$.

In this problem, the displacement vector is $\langle 6-0, 12-\cancel{10}, 20-8 \rangle = \langle 6, 2, 12 \rangle$. Therefore

$$w = \vec{F} \cdot \vec{D} = \langle 8, -6, 9 \rangle \cdot \langle 6, 2, 12 \rangle =$$
$$8(6) + (-6)2 + 9(12) = 48 - 12 + 108 = 144 \text{ joules.}$$

7. Here, we do not have the vectors $\vec{F}$ and $\vec{D}$, but we have $|\vec{D}|$ and $|\vec{F}|$, thus we can use the equation

$$w = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos\theta$$

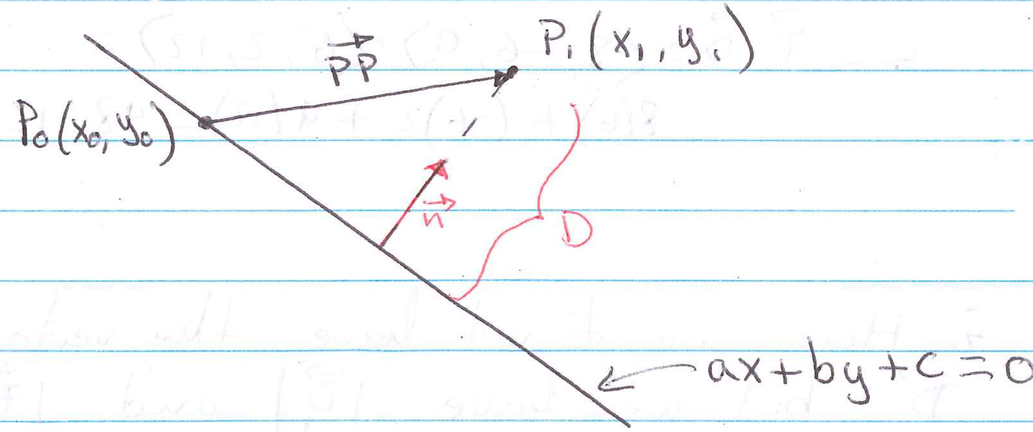
In this case:

$$w = (30)(80) \cos 40^\circ \approx 1839 \text{ ft-lb.}$$

8. Here we have the same situation as in the previous problem.

$$w = |\vec{F}||\vec{D}|\cos\theta$$
$$= (400)(120)\cos 36^{\circ} \approx 38833 \text{ ft-lb.}$$

9. Consider the following figure.



We note that the distance from the point P_1 to the line, denoted by D, is the absolute value of the scalar projection of the vector $\vec{PP}$, which goes from a point P_0 on the line to P_1, onto the vector $\vec{n}$, which is perpendicular to the line.

$$D = |\text{comp}_{\vec{n}} \vec{PP}|$$

We know that

$$\frac{\Delta y}{\Delta x} = \frac{b}{a} \quad (1) \qquad \text{and} \qquad (\Delta x)^2 + (\Delta y)^2 = 1 \quad (2)$$

From (1):

$$\Delta y = \frac{b}{a} \Delta x \quad (3)$$

Substituting the value of Δy in (2)

$$(\Delta x)^2 + \left(\frac{b}{a} \Delta x\right)^2 = 1$$

$$(\Delta x)^2 + \frac{b^2}{a^2} (\Delta x)^2 = 1$$

$$(\Delta x)^2 \left(1 + \frac{b^2}{a^2}\right) = 1$$

$$(\Delta x)^2 = \frac{1}{1 + \frac{b^2}{a^2}} = \frac{1}{\frac{a^2 + b^2}{a^2}} = \frac{a^2}{a^2 + b^2}$$

$$\Delta x = \frac{a}{\sqrt{a^2 + b^2}}$$

Substituting Δx in ③

$$\Delta y = \frac{b}{a}\left(\frac{a}{\sqrt{a^2+b^2}}\right) = \frac{b}{\sqrt{a^2+b^2}}$$

$\therefore$ the vector $\vec{n} = \left\langle \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right\rangle$

Now, computing the scalar projection we obtain

$$\left|\text{comp}_{\vec{n}} \overrightarrow{PP}\right| = \frac{\left|\vec{n} \cdot \overrightarrow{PP}\right|}{|\vec{n}|} = \left|\vec{n} \cdot \overrightarrow{PP}\right| \left(\text{because } |\vec{n}| = 1\right)$$

$\overrightarrow{PP} = \langle x_1 - x_0, y_1 - y_0 \rangle$, thus

$$\left|\vec{n} \cdot \overrightarrow{PP}\right| = \left|\left\langle \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right\rangle \cdot \langle x_1 - x_0, y_1 - y_0 \rangle\right|$$

$$= \left|\frac{1}{\sqrt{a^2+b^2}}\left(a(x_1 - x_0) + b(y_1 - y_0)\right)\right|$$

$$= \frac{1}{\sqrt{a^2+b^2}}\left|ax_1 - ax_0 + by_1 - by_0\right| =$$

$$= \frac{1}{\sqrt{a^2+b^2}} \left| ax_1 + by_1 - ax_0 - by_0 \right|$$

But since P_0 is on the line, ~~[scribbled out]~~ it must satisfy

$$ax_0 + by_0 + c = 0$$
$$c = -ax_0 - by_0$$

Thus

$$\boxed{D = \frac{1}{\sqrt{a^2+b^2}} \left| ax_1 + by_1 + c \right|}$$

Using this formula in the example:

$$D = \frac{1}{\sqrt{3^2+(-4)^2}} \left| 3(-2) + (-4)(3) + 5 \right|$$

$$= \frac{1}{\sqrt{25}} \left| -6 - 12 + 5 \right| =$$

$$= \frac{1}{5} \left| -13 \right| = \boxed{\frac{13}{5}}$$

10. Property 2: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$\vec{a} \cdot \vec{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle =$
$a_1 b_1 + a_2 b_2 + a_3 b_3 = b_1 a_1 + b_2 a_2 + b_3 a_3 =$
$\langle b_1, b_2, b_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = \vec{b} \cdot \vec{a}$

Property 4: $(\alpha \vec{a}) \cdot \vec{b} = \alpha (\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\alpha \vec{b})$

$(\alpha \vec{a}) \cdot \vec{b} = (\alpha \langle a_1, a_2, a_3 \rangle) \cdot \langle b_1, b_2, b_3 \rangle =$
$\langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle =$
$\alpha a_1 b_1 + \alpha a_2 b_2 + \alpha a_3 b_3 = \alpha (a_1 b_1 + a_2 b_2 + a_3 b_3) =$
$\alpha (\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle) = \alpha (\vec{a} \cdot \vec{b})$.

From: $(\alpha \vec{a}) \cdot \vec{b} = \alpha a_1 b_1 + \alpha a_2 b_2 + \alpha a_3 b_3$
we note that it is equal to
$a_1 (\alpha b_1) + a_2 (\alpha b_2) + a_3 (\alpha b_3) = \langle a_1, a_2, a_3 \rangle \cdot$
$\langle \alpha b_1, \alpha b_2, \alpha b_3 \rangle = \vec{a} \cdot (\alpha \vec{b})$.

Property 5: $\vec{0} \cdot \vec{a} = 0$

$\vec{0} \cdot \vec{a} = \langle 0,0,0 \rangle \cdot \langle a_1, a_2, a_3 \rangle = 0a_1 + 0a_2 + 0a_3$
$= 0$