Homework #5

1. $\vec{a} = \langle 6, 0, -2 \rangle$ and $\vec{b} = \langle 0, 8, 0 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 0 & -2 \\ 0 & 8 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 8 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} 6 & -2 \\ 0 & 0 \end{vmatrix} \hat{j} + \begin{vmatrix} 6 & 0 \\ 0 & 8 \end{vmatrix} \hat{k} =$$

$$(0 - (-2)(8))\hat{i} - (6(0) - (-2)(0))\hat{j} + (6(8) - 0)\hat{k} =$$

$$\boxed{16\hat{i} + 48\hat{k}}$$

To verify that $\vec{c} = \langle 16, 0, 48 \rangle$ is perpendicular to $\vec{a}$ and $\vec{b}$, we need to verify that $\underbrace{(\vec{a} \times \vec{b})}_{\vec{c}} \cdot \vec{a} = 0$ and

$(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle 16, 0, 48 \rangle \cdot \langle 6, 0, -2 \rangle = 96 + 0 - 96 = 0$$

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = \langle 16, 0, 48 \rangle \cdot \langle 0, 8, 0 \rangle = 0 + 0 + 0 = 0$$

2. $\vec{a} = \langle t, t^2, t^3 \rangle$ and $\vec{b} = \langle 1, 2t, 3t^2 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & t^2 & t^3 \\ 1 & 2t & 3t^2 \end{vmatrix} = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} \hat{i} - \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} \hat{j} + \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} \hat{k} =$$

$$\left(t^2(3t^2) - (2t)(t^3)\right)\hat{i} - \left(3t^2 \cdot t - t^3\right)\hat{j} + \left(2t \cdot t - t^2\right)\hat{k} =$$

$$(3t^4 - 2t^4)\hat{i} - (3t^3 - t^3)\hat{j} + (2t^2 - t^2)\hat{k} =$$

$$\boxed{t^4\hat{i} - 2t^3\hat{j} + t^2\hat{k}}$$

Let's verify orthogonality:

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle t^4, -2t^3, t^2 \rangle \cdot \langle t, t^2, t^3 \rangle =$$

$$t^5 - 2t^5 + t^5 = 0$$

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = \langle t^4, -2t^3, t^2 \rangle \cdot \langle 1, 2t, 3t^2 \rangle =$$

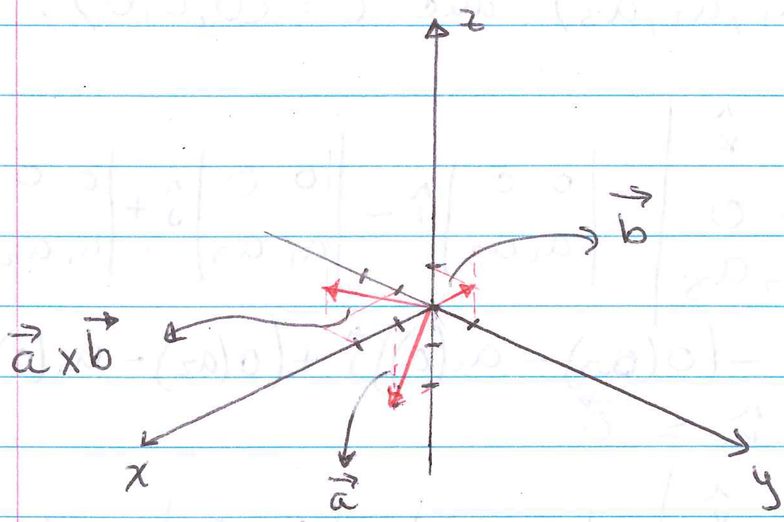
$$t^4 - 4t^4 + 3t^4 = 0$$

3. $\vec{a} = \hat{i} - 2\hat{k}$, $\vec{b} = \hat{j} + \hat{k}$

$$\vec{a} \times \vec{b} = \left|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{matrix}\right| \begin{matrix} \hat{i} & \hat{j} \\ 1 & 0 \\ 0 & 1 \end{matrix}$$

$$\begin{matrix} - & - & - \\ + & + & + \end{matrix}$$

$$\vec{a} \times \vec{b} = 0(1)\hat{i} + -2(0)\hat{j} + 1(1)\hat{k} - 0(0)\hat{k} - 1(-2)\hat{i} - 1(1)\hat{j} = \hat{k} + 2\hat{i} - \hat{j} = \boxed{2\hat{i} - \hat{j} + \hat{k}}$$



4. $\vec{a} = \langle 1, 2, 1 \rangle$, $\vec{b} = \langle 0, 1, 3 \rangle$

$$\vec{a} \times \vec{b} = \left|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{matrix}\right| = \left|\begin{matrix} 2 & 1 \\ 1 & 3 \end{matrix}\right| \hat{i} - \left|\begin{matrix} 1 & 1 \\ 0 & 3 \end{matrix}\right| \hat{j} + \left|\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}\right| \hat{k} =$$

$$\big(2(3) - (1)1\big)\hat{i} - \big(1(3) - 0(1)\big)\hat{j} + \big(1(1) - 0(2)\big)\hat{k} =$$

$$\boxed{5\hat{i} - 3\hat{j} + \hat{k}}$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \hat{i} - \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} \hat{j} + \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \hat{k} =$$

$$(1(1) - 2(3))\hat{i} - (0(1) - 1(3))\hat{j} + (0(2) - 1(1))\hat{k} =$$

$$\boxed{-5\hat{i} + 3\hat{j} - \hat{k}}$$

Note that $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

5. Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{0} = \langle 0,0,0 \rangle$, then

$$\vec{0} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ a_2 & a_3 \end{vmatrix} \hat{i} - \begin{vmatrix} 0 & 0 \\ a_1 & a_3 \end{vmatrix} \hat{j} + \begin{vmatrix} 0 & 0 \\ a_1 & a_2 \end{vmatrix} \hat{k} =$$

$$(0(a_3) - a_2(0))\hat{i} - (0(a_3) - a_1(0))\hat{j} + (0(a_2) - a_1(0))\hat{k} =$$

$$0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

$$\vec{a} \times \vec{0} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ 0 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ 0 & 0 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix} \hat{k} =$$

$$(a_2(0) - 0(a_3))\hat{i} - (a_1(0) - 0(a_3))\hat{j} + (a_1(0) - 0(a_2))\hat{k} =$$

$$0\hat{i} + 0\hat{j} + 0\hat{k} = \langle 0,0,0 \rangle = \vec{0}$$

6. The volume of the parallelepiped with adjacent edges PQ, PR, and PS is equal to

$$\left|\overrightarrow{PQ} \cdot \left(\overrightarrow{PR} \times \overrightarrow{PS}\right)\right|$$

$\overrightarrow{PQ} = \langle 4-2, 1-0, 0-(-1) \rangle = \langle 2, 1, 1 \rangle$
$\overrightarrow{PR} = \langle 3-2, -1-0, 1-(-1) \rangle = \langle 1, -1, 2 \rangle$
$\overrightarrow{PS} = \langle 2-2, -2-0, -2-(-1) \rangle = \langle 0, -2, -1 \rangle$

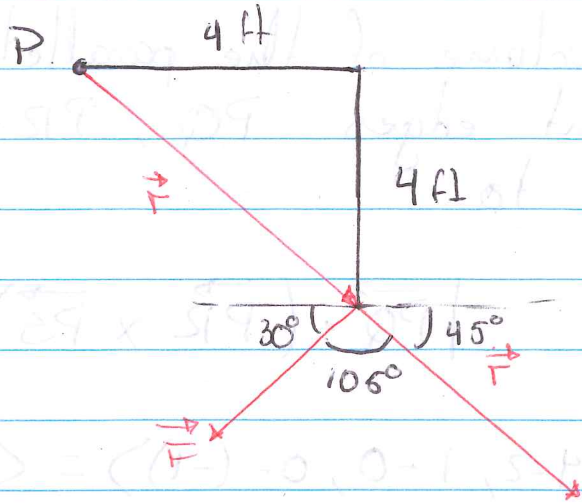
$$\overrightarrow{PR} \times \overrightarrow{PS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 0 & -2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -2 & -1 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} \hat{k} =$$

$$(-1(-1) - (-2)2)\hat{i} - (1(-1) - 0(2))\hat{j} + (1(-2) - 0(-1))\hat{k} =$$
$$5\hat{i} + \hat{j} - 2\hat{k} = \langle 5, 1, -2 \rangle$$

$$\left|\overrightarrow{PQ} \cdot \left(\overrightarrow{PR} \times \overrightarrow{PS}\right)\right| = \left|\langle 2, 1, 1 \rangle \cdot \langle 5, 1, -2 \rangle\right| =$$
$$2(5) + 1(1) + 1(-2) = 10 + 1 - 2 = 9$$

$\therefore$ Volume of parallelepiped $= 9$ cubic units

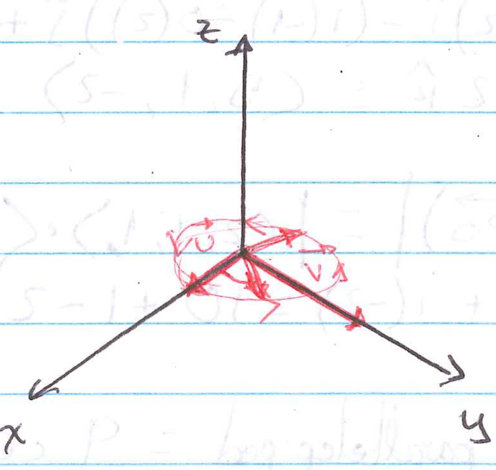
7.



The magnitude of the torque about P is

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}|\,|\vec{F}| \sin\theta = \left(\sqrt{4^2+4^2}\right)(36)\sin 105^\circ = \left(\sqrt{32}\right)(36)\sin 105^\circ \approx 197 \text{ ft-lb.}$$

8.

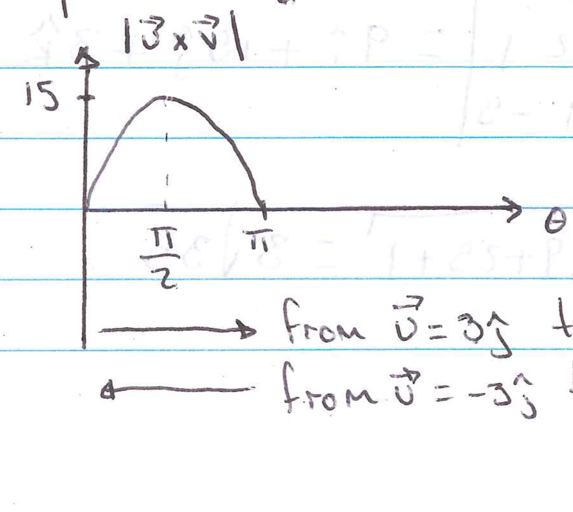


$\vec{v} = 5\hat{j}$, $|\vec{u}| = 3$

Suppose $\vec{u}$ starts it's rotation at $3\hat{j}$, then $|\vec{u} \times \vec{v}| = 0$ because $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$ for $0 \leq \theta \leq \pi$, but in this case $\sin\theta = 0$.

As $\vec{u}$ rotates counterclockwise $\vec{u} \times \vec{v}$ points downward due to the right hand rule. When $\vec{u} = -3\hat{i}$ the magnitude of $\vec{u} \times \vec{v}$ is maximum because $\sin\frac{\pi}{2} = 1$. This maximum is $|\vec{u}||\vec{v}| = 3 \cdot 5 = 15$. As $\vec{u}$ keeps rotating, $|\vec{u} \times \vec{v}|$ decreases until it becomes zero again when $\vec{u} = -3\hat{j}$.

Then, $\vec{u} \times \vec{v}$ switches orientation and starts pointing in the positive z-direction. When $\vec{u} = 3\hat{i}$, the magnitude of $\vec{u} \times \vec{v}$ reaches another maximum. Then it decreases again until $\vec{u}$ reaches its initial position.



9. $\vec{a} = \langle 1, 1, -2 \rangle$ $\vec{b} = \langle 3, -2, 1 \rangle$, $\vec{c} = \langle 0, 1, -5 \rangle$

- $2\vec{a} + 3\vec{b} = 2\langle 1, 1, -2 \rangle + 3\langle 3, -2, 1 \rangle =$
$\langle 2, 2, -4 \rangle + \langle 9, -6, 3 \rangle =$
$\langle 2+9, 2-6, -4+3 \rangle = \langle 11, -4, -1 \rangle$

- $|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$

- $\vec{a} \cdot \vec{b} = \langle 1, 1, -2 \rangle \cdot \langle 3, -2, 1 \rangle = 1(3) + 1(-2) + (-2)(1) =$
$3 - 2 - 2 = -1$

- $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \hat{j} +$

$\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} \hat{k} = (1(1) - (-2)(-2))\hat{i} - (1(1) - 3(-2))\hat{j} + (-2(1) - 3(1))$

$= -3\hat{i} - 7\hat{j} - 5\hat{k}$

- $|\vec{b} \times \vec{c}|$:

$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 9\hat{i} + 15\hat{j} + 3\hat{k}$

$|\vec{b} \times \vec{c}| = 3\sqrt{9+25+1} = 3\sqrt{35}$

- $\vec{a} \cdot (\vec{b} \times \vec{c}) = \langle 1, 1, -2 \rangle \cdot \langle 9, 15, 3 \rangle = 9 + 15 - 6 = 18$

- $\vec{c} \times \vec{c}$

$\vec{c} \times \vec{c} = \vec{0}$, because $\vec{c}$ is parallel to itself.

- $\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 9 & 15 & 3 \end{vmatrix} = 33\hat{i} - 21\hat{j} + 6\hat{k}$

- $\text{comp}_{\vec{a}} \vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

$|\vec{a}| = \sqrt{1^2 + 1^2 + 4} = \sqrt{6}$

$$\text{comp}_{\vec{a}} \vec{b} = \frac{1}{\sqrt{6}} (\langle 1, 1, -2 \rangle \cdot \langle 3, -2, 1 \rangle)$$

$$= \frac{1}{\sqrt{6}} (3 + (-2) + (-2)) = -\frac{1}{\sqrt{6}}$$

- $\text{proj}_{\vec{a}} \vec{b} = (\text{comp}_{\vec{a}} \vec{b}) \dfrac{\vec{a}}{|\vec{a}|} = -\dfrac{1}{\sqrt{6}} \cdot \dfrac{1}{\sqrt{6}} \langle 1, 1, -2 \rangle$

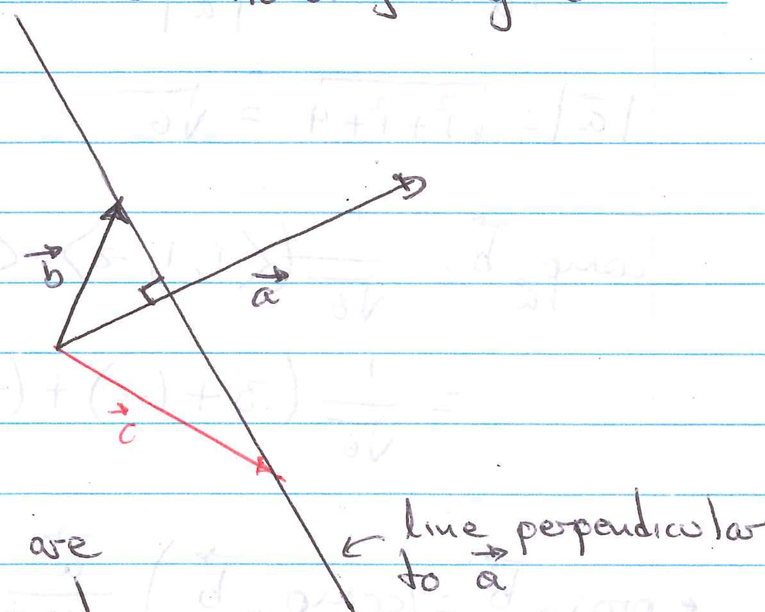
$$= -\frac{1}{6} \langle 1, 1, -2 \rangle$$

- Angle between $\vec{a}$ and $\vec{b}$

$$\theta = \arccos\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right) = \arccos\left(\frac{-1}{\sqrt{6}\sqrt{14}}\right) =$$

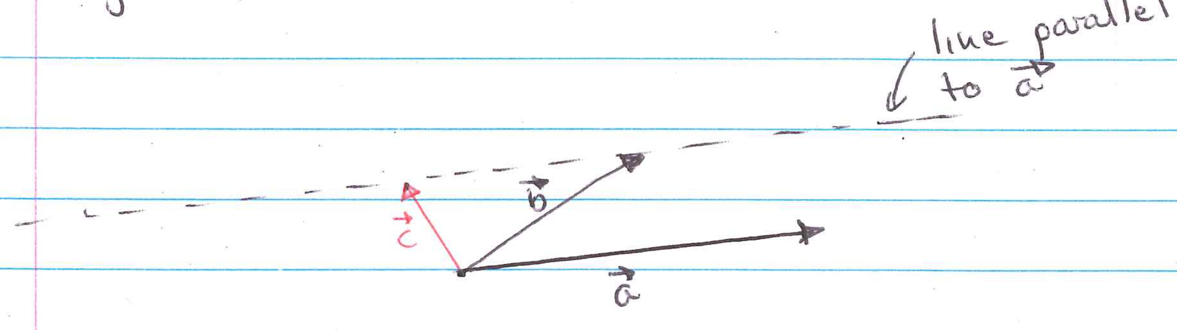
$$\arccos\left(\frac{-1}{\sqrt{84}}\right) \approx 96^\circ$$

- If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, does not mean that $\vec{b} = \vec{c}$, consider the following figure



Vectors $\vec{b}$ and $\vec{c}$ are very different, yet $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$

If $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ doesn't mean that $\vec{b} = \vec{c}$, consider the following figure:



Vectors $\vec{b}$ and $\vec{c}$ are different, yet $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$

If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ can only mean that $\vec{b} = \vec{c}$. The idea is that only one vector can have the same scalar projection onto $\vec{a}$ and span the same orthogonal vector: