Homework #6

1. $P_1(0,0,0)$ $P_2(1,2,3)$

First, we find a direction vector of the line:

$$\vec{v} = \langle 1-0, 2-0, 3-0\rangle = \langle 1,2,3\rangle$$

The reference point is chosen to be $\langle 0,0,0\rangle$. Then, the vector equation is

$$\vec{r}(t) = \langle 0,0,0\rangle + t\langle 1,2,3\rangle = \langle t, 2t, 3t\rangle$$

Parametric equations: $x=t$, $y=2t$, $z=3t$
Symmetric equations:

$$x = \frac{y}{2} = \frac{z}{3}$$

2. From $\vec{a} = \hat{i}+\hat{j}$ and $\vec{b} = \hat{j}+\hat{k}$ we can find a vector perpendicular to both of them and we are going to use it as direction vector of the line whose equations we are looking for:

$$\vec{v} = \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \hat{i} - \hat{j} + \hat{k} = \langle 1,-1,1\rangle$$

The vector equation of the line is

$$\vec{r}(t) = \langle 2,1,0\rangle + t\langle 1,-1,1\rangle$$
$$= \langle 2+t, 1-t, t\rangle$$

Parametric equations: $x = 2+t$, $y = 1-t$, $z = t$

Symmetric equations:

$$x-2 = 1-y = z$$

3. We need a point on the plane and its normal vector.

The point can be found from the line $x = 3+2t$, $y = t$, $z = 8-t$. If we set $t=0$, we obtain the point $(3,0,8)$.

The normal vector can be $\langle 2,4,8\rangle$ because the plane whose equation we are trying to find is parallel to the plane $2x+4y+8z=17$. Therefore their normal vectors must be parallel.

The equation of the plane is then

$$(\vec{r}-\vec{r}_0)\cdot\vec{n}=0$$

$$(\langle x,y,z\rangle-\langle 3,0,8\rangle)\cdot\langle 2,4,8\rangle=0$$
$$\langle x-3,y,z-8\rangle\cdot\langle 2,4,8\rangle=0$$
$$2(x-3)+4y+8(z-8)=0$$
$$2x+4y+8z-6-64=0$$
$$\underline{2x+4y+8z-70=0}$$

4. In order to find two more points on the plane, in addition to the point already given (1,2,3), we can use the line. If, for instance, we set $t=0$ and $t=1$, we get the points (0,1,2) and (3,2,1).

Two vectors on the plane are then

$$\vec{a}=\langle 0-1,1-2,2-3\rangle=\langle -1,-1,-1\rangle$$
$$\vec{b}=\langle 3-1,2-2,1-3\rangle=\langle 2,0,-2\rangle$$

The normal vector of the plane is

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & -1 \\ 2 & 0 & -2 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 0 & -2 \end{vmatrix} \hat{i} - \begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix} \hat{j} +$$

$$\begin{vmatrix} -1 & -1 \\ 2 & 0 \end{vmatrix} \hat{k} = ((-1)(-2) - 0)\hat{i} - ((-1)(-2) - (2)(-1))\hat{j} +$$

$$(0 - (-1)(2))\hat{k} = 2\hat{i} - 4\hat{j} + 2\hat{k} =$$

$$\langle 2, -4, 2 \rangle$$

The equation of the plane then is

$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

$$(\langle x, y, z \rangle - \langle 1, 2, 3 \rangle) \cdot \langle 2, -4, 2 \rangle = 0$$

$$\langle x-1, y-2, z-3 \rangle \cdot \langle 2, -4, 2 \rangle = 0$$

$$2(x-1) + (-4)(y-2) + 2(z-3) = 0$$

$$2x - 4y + 2z - 2 + 8 - 6 = 0$$

$$2x - 4y + 2z = 0$$

5. To find the point at which the line $x=1+2t$, $y=4t$, $z=2-3t$ intersects the plane $x+2y-z+1=0$, we first find the value of t at which the line "touches" the plane. This is done by substituting the parametric equations in the plane's equation:

$$(1+2t)+2(4t)-(2-3t)+1=0$$
$$1+2t+8t-2+3t+1=0$$

$13t=0$, which means that at $t=0$, the line touches the plane.

It is a matter of setting $t=0$ in the parametric equations to find the coordinates of the intersection point:

$$x=1,\ y=0,\ z=2$$

$(1,0,2)$ is the intersection point.

6. The angle between two planes is equal to the angle between their normal vectors if they form an acute angle.

The normal vectors of the planes involved are:

$$\vec{n}_1 = \langle 1, 2, 2 \rangle \quad \text{and} \quad \vec{n}_2 = \langle 2, 1, -3 \rangle$$

the angle between these vectors can be found using their dot product:

$$\theta = \arccos\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}\right) =$$

$$\arccos\left(\frac{1(2) + (2)1 + 2(-3)}{\sqrt{1+4+4}\,\sqrt{4+1+9}}\right) =$$

$$\arccos\left(\frac{-2}{\sqrt{9}\,\sqrt{14}}\right) =$$

$$\arccos\left(\frac{-2}{3\sqrt{14}}\right) = \arccos(-0.178)$$

$\theta \approx 100.3°$. The angle between the planes is thus $180 - 100.3 = \underline{79.7°}$

7. The intersection line of two planes can be found by determining a point on the line and calculating its direction vector.

The first requirement, (a point on the line) is solved if we "cut" the planes with a third plane, say $y=0$. The planes' equations reduce to

$$3x + z = 1 \quad (1)$$
$$2x - 3z = 3 \quad (2)$$

From (1), $z = 1 - 3x$

z in (2): $2x - 3(1-3x) = 3$

$$2x - 3 + 9x = 3$$
$$11x = 6 \Rightarrow x = \frac{6}{11}$$

x in $z = 1-3x$, we get $z = 1 - 3\left(\frac{6}{11}\right) =$

$$\frac{11}{11} - \frac{18}{11} = -\frac{7}{11}.$$

The point $\left(\frac{6}{11}, 0, -\frac{7}{11}\right)$ is on the intersection line.

The direction vector of the line is perpendicular to the normal vectors of the planes. So

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle 3, -2, 1 \rangle \times \langle 2, 1, -3 \rangle =$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 2 & 1 & -3 \end{vmatrix} = 5\hat{i} + 11\hat{j} + 7\hat{k}$$

The vector equation of the line is:

$$\vec{r}(t) = \langle \tfrac{6}{11}, 0, -\tfrac{7}{11} \rangle + t\langle 5, 11, 7 \rangle$$

$$= \langle \tfrac{6}{11} + 5t, 11t, -\tfrac{7}{11} + 7t \rangle$$

The parametric equations of the line are

$$\underline{x = \frac{6}{11} + 5t, \quad y = 11t, \quad z = -\frac{7}{11} + 7t}$$

The angle between the planes is

$$\theta = \arccos\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}\right)$$

$$\theta = \arccos\left(\frac{3(2)+(-2)1+1(-3)}{\sqrt{9+4+1}\ \sqrt{4+1+9}}\right) =$$

$$= \arccos\left(\frac{1}{14}\right) = \boxed{85.9^\circ}$$

8. The plane passes through $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$. We can find two vectors:

$$\vec{a} = \langle 0-a,\ b-0,\ 0-0\rangle = \langle -a, b, 0\rangle$$
$$\vec{b} = \langle 0-a,\ 0-0,\ c-0\rangle = \langle -a, 0, c\rangle$$

The normal vector of this plane is

$$\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = bc\,\hat{i} + ac\,\hat{j} + ab\,\hat{k}$$

Then $(\vec{r}-\vec{r}_0)\cdot\vec{n} = 0$

$$\left(\langle x,y,z\rangle - \langle a,0,0\rangle\right)\cdot\langle bc, ac, ab\rangle = 0$$

$$bc(x-a) + acy + abz = 0$$

$$\boxed{bcx + acy + abz - abc = 0}$$

If $a, b, c \neq 0$, we can divide by abc giving

$$\boxed{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1}$$

9. The distance from a point to a plane is equal to $|\text{comp}_{\vec{n}} \vec{PP}|$, where $\vec{n}$ is a normal vector to the plane and $\vec{PP}$ is a vector "linking" a point on the plane with the point given. Thus

$\vec{n} = \langle 1, -2, -4 \rangle$

A point on the plane is $(8, 0, 0)$. This point is found by setting y and z to zero in the plane's equation.

$$|\text{comp}_{\vec{n}} \vec{PP}| = \frac{|\vec{n} \cdot \vec{PP}|}{|\vec{n}|} = \frac{|\langle 1, -2, -4 \rangle \cdot \langle 0-8, 1-0, 3-0 \rangle|}{\sqrt{1 + 4 + 16}}$$

$$= \left|\frac{-8 - 2 - 12}{\sqrt{21}}\right| = \boxed{\frac{22}{\sqrt{21}}}$$

10. ① $x = y = z$, ② $x + 1 = \frac{y}{2} = \frac{z}{3}$

If the lines intersect, then there must be a point that satisfies ① and ②.

If we take from ① $x = y$ and use this in ② we get

$$x + 1 = \frac{y}{2} = \frac{x}{2} \text{ (by ①)}$$

$$2x + 2 = x \Rightarrow 2x - x = -2$$
$$x = -2$$

By ① this means that $x = y = z = -2$. However, from ②

$$\frac{y}{2} = \frac{z}{3}$$

$$\frac{-2}{2} = \frac{-2}{3} \leftarrow \text{a contradiction}$$

Therefore, the assumption that the lines intersect is incorrect. This means that the lines are skew lines.

The distance between the lines is given by $|comp_{\vec{n}} \vec{PP}|$, where $\vec{n}$ is a vector perpendicular to the lines' direction vectors and $\vec{PP}$ is a vector connecting a point on one line with a point on the other line.

The direction vectors of the lines are:

$\vec{v}_1 = \langle 1, 1, 1 \rangle$

$\vec{v}_2 = \langle 1, 2, 3 \rangle$ (taken from the denominators of the symmetric equations.)

Thus $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \hat{i} - 2\hat{j} + \hat{k}$

A point on line 1 is $(1, 1, 1)$. A point on line 2 is $(-1, 0, 0)$. The vector $\vec{PP}$ is therefore $\langle 1-(-1), 1-0, 1-0 \rangle = \langle 2, 1, 1 \rangle$.

$$|comp_{\vec{n}} \vec{PP}| = \frac{|\vec{n} \cdot \vec{PP}|}{\vec{n}} = \frac{|\langle 1, -2, 1 \rangle \cdot \langle 2, 1, 1 \rangle|}{\sqrt{1+4+1}} = \frac{|2-2+1|}{\sqrt{6}} = \boxed{\frac{1}{\sqrt{6}}}$$