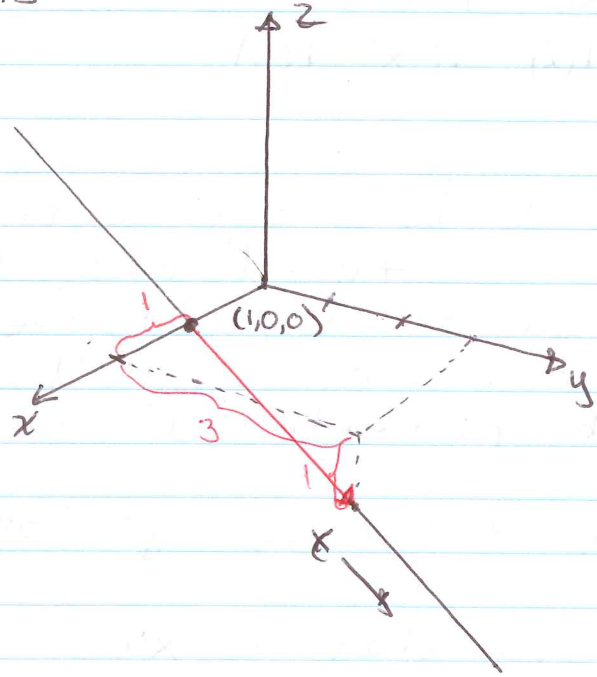
Homework 8

1. $\vec{r}(t) = \langle 1+t, 3t, -t \rangle$ can be rewritten as $\vec{r}(t) = \langle 1,0,0 \rangle + t\langle 1, 3, -1 \rangle$, which is the vector equation of a line through $(1,0,0)$ and direction vector $\langle 1,3,-1 \rangle$. So, a sketch of the curve represented by $\vec{r}(t)$ is



2. The parametric equations of the helix are $x = \sin(t)$, $y = \cos(t)$, $z = t$. Then the points of intersection of the helix with the sphere $x^2+y^2+z^2=5$ should be those that satisfy:

$$\underbrace{(\sin(t))^2 + (\cos(t))^2}_{1} + t^2 = 5$$

So $\quad 1 + t^2 = 5 \quad$ or $\quad t = \pm 2$

The points of intersection are then

$(\sin(2), \cos(2), 2)$ and

$(-\sin(2), \cos(2), -2)$

odd function $f(-x) = -f(x)$ even function $f(-x) = f(x)$

3. Let C be the curve of intersection between $x^2 + y^2 = 4$ and $z = xy$. The projection of C onto the xy-plane is the circle $x^2 + y^2 = 4$, which can be parametrized as

$$x = 2\cos\theta \quad , \quad y = 2\sin\theta.$$

Then given that $z = xy$, we have

$$\begin{aligned} z = (2\cos\theta)(2\sin\theta) &= 4\sin\theta\cos\theta \\ &= 2(2\sin\theta\cos\theta) \\ &= 2\sin(2\theta) \end{aligned}$$

The curve C is then represented by

$$\vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 2\sin(2\theta) \rangle$$

with $0 \leq \theta \leq 2\pi$

4. The tangent line to the curve represented by $\vec{r}(t) = \langle \ln(t), 2\sqrt{t}, t^2 \rangle$ at $(0,2,1)$ has the same direction as $\vec{r}\,'(t)$ at $(0,2,1)$. Therefore the direction vector of the tangent line is

$$\vec{r}\,'(t) = \left\langle \frac{1}{t}, \frac{1}{\sqrt{t}}, 2t \right\rangle \quad @\ t=1$$

$$\vec{r}\,'(1) = \langle 1, 1, 2 \rangle$$

Thus, the vector equation of the tangent line is

$$\vec{r}(s) = \langle 0,2,1 \rangle + s\langle 1,1,2 \rangle$$
$$= \langle s, 2+s, 1+2s \rangle$$

The parametric equations are:

$$x = s, \quad y = 2+s, \quad z = 1+2s$$

5. The point of intersection of the curves represented by $\vec{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle$ and $\vec{r}_2(s) = \langle 3-s, s-2, s^2 \rangle$ is the point (x,y,z) such that

$$x = t = 3-s \quad (1)$$
$$y = 1-t = s-2 \quad (2)$$
$$z = 3+t^2 = s^2 \quad (3)$$

(1) in (3): $3 + (3-s)^2 = s^2$

$3 + 9 - 6s + s^2 = s^2$

$12 - 6s = 0 \Rightarrow s = 2$

Back in ①: $t = 3 - 2 = 1$

The point of intersection is then

$(1, 0, 4)$

The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection.

$\vec{r}_1'(t) = \langle 1, -1, 2t \rangle \Rightarrow \vec{r}_1'(1) = \langle 1, -1, 2 \rangle.$

$\vec{r}_2'(s) = \langle -1, 1, 2s \rangle \Rightarrow \vec{r}_2'(2) = \langle -1, 1, 4 \rangle$

$$\frac{\vec{r}_1'(1) \cdot \vec{r}_2'(2)}{|\vec{r}_1'(1)|\,|\vec{r}_2'(2)|} = \cos\theta = \frac{-1-1+8}{\sqrt{6}\sqrt{18}} = \frac{6}{\sqrt{6}\sqrt{18}}$$

$$\theta = \arccos\left(\frac{6}{\sqrt{6}\sqrt{18}}\right) = \arccos\left(\frac{1}{\sqrt{3}}\right)$$

6. $$\vec{r}(t) = \int \vec{r}\,'(t)\,dt$$

$$= \int \langle t, e^t, te^t \rangle\,dt$$

$$= \left\langle \int t\,dt, \int e^t\,dt, \int te^t\,dt \right\rangle$$

$$= \left\langle \frac{t^2}{2}, e^t, (t-1)e^t \right\rangle + \vec{c}$$

Since $\vec{r}(0) = \langle 1, 1, 1 \rangle$ we have

$\langle 0, 1, -1 \rangle + \vec{c} = \langle 1, 1, 1 \rangle \therefore \vec{c} = \langle 1, 0, 2 \rangle$

Therefore

$$\vec{r}(t) = \left\langle \frac{t^2}{2} + 1, e^t, (t-1)e^t + 2 \right\rangle$$

7. The arc length is given by

$$L = \int_0^{\frac{\pi}{4}} |\vec{r}\,'(t)|\,dt$$

Let us first find $|\vec{r}\,'(t)|$.

$\vec{r}\,'(t) = \langle -\sin t, \cos t, -\tan t \rangle$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$$

$$= \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = |\sec t|$$

Since $\sec t > 0$ for $0 \le t \le \frac{\pi}{4}$ we can write

$$|\vec{r}\,'(t)| = \sec t$$

$$\therefore \quad L = \int_0^{\pi/4} \sec t \, dt = \Big[\ln |\sec t + \tan t| \Big]_0^{\pi/4}$$

$$L = \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$

$$= \ln|\sqrt{2}+1| - \ln|1+0| = \ln(\sqrt{2}+1).$$

8. Let's find first $\vec{T}(t)$, which means that we need to know $\vec{r}\,'(t)$ and its norm.

$\vec{r}(t) = \langle t, t^2/2, t^2 \rangle$, thus

$$\vec{r}\,'(t) = \langle 1, t, 2t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{1+t^2+4t^2} = \sqrt{1+5t^2}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{\sqrt{1+5t^2}} \langle 1, t, 2t \rangle$$

Then, using $\frac{d}{dt}[f(t)\vec{r}(t)] = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$

we have

$$\vec{T}\,'(t) = \frac{-1}{2}(1+5t^2)^{-3/2}(10t)\langle 1, t, 2t \rangle + \frac{1}{\sqrt{1+5t^2}} \langle 0, 1, 2 \rangle$$

After simplification:

$$\vec{T}'(t) = \frac{1}{\sqrt{1+5t^2}} \left\langle \frac{-5t}{1+5t^2}, \frac{1}{1+5t^2}, \frac{2}{1+5t^2} \right\rangle$$

$$= \frac{1}{(1+5t^2)^{3/2}} \langle -5t, 1, 2 \rangle$$

$$|\vec{T}'(t)| = \frac{1}{(1+5t^2)^{3/2}} \sqrt{25t^2+1+4} = \frac{\sqrt{25t^2+5}}{\sqrt{(1+5t^2)^3}} =$$

$$\sqrt{\frac{5(1+5t^2)}{(1+5t^2)^3}} = \frac{\sqrt{5}}{(1+5t^2)}$$

Therefore

$$K(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\frac{\sqrt{5}}{1+5t^2}}{\sqrt{1+5t^2}} = \boxed{\frac{\sqrt{5}}{(1+5t^2)^{3/2}}}$$

9. A parabola that passes through the origin has a general equation $y = Ax^2$.
Let's use $K(x) = \frac{|\vec{r}'(x) \times \vec{r}''(x)|}{|\vec{r}'(x)|^3}$ to find the curvature of $y = Ax^2$ when $x = 0$.

The vector equation that describes the parabola is $\vec{r}(x) = \langle x, Ax^2, 0 \rangle$. Then, $\vec{r}'(x) = \langle 1, 2Ax, 0 \rangle$ and $\vec{r}''(x) = \langle 0, 2A, 0 \rangle$

$$\vec{r}'(x) \times \vec{r}''(x) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2Ax & 0 \\ 1 & 2A & 0 \end{vmatrix} = 2A\,\hat{k}$$

$$|\vec{r}'(x) \times \vec{r}''(x)| = 2A$$

Now, $|\vec{r}'(x)| = \sqrt{1 + (2Ax)^2}$

$$\kappa(x) = \frac{|\vec{r}'(x) \times \vec{r}''(x)|}{|\vec{r}'(x)|^3} = \frac{2A}{\left(\sqrt{1+(2Ax)^2}\right)^3}$$

$\kappa(0) = 2A = 4 \Leftarrow$ Given.

Thus
$A = 2$ and the parabola that has curvature 4 at the origin is
$\underline{y = 2x^2}$

10. Let's write $f(t)$ simply as f and $g(t)$ as g. Then $\vec{r} = \langle f, g, 0 \rangle$ and $\vec{r}' = \langle f', g', 0 \rangle$ and $\vec{r}'' = \langle f'', g'', 0 \rangle$.

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f' & g' & 0 \\ f'' & g'' & 0 \end{vmatrix} = (f'g'' - f''g')\,\hat{k}$$

$$|\vec{r}' \times \vec{r}''| = f'g'' - f''g'$$

$$|\vec{r}'| = \sqrt{(f')^2 + (g')^2}$$

Thus

$$\kappa = \frac{f'g'' - f''g'}{\left((f')^2 + (g')^2\right)^{3/2}}$$