

Homework 9

1. A vector function that describes $y=f(x)$ in the xy -plane is $\vec{r}(x)=\langle x, f(x), 0 \rangle$.

Then, $\vec{r}'(x)=\langle 1, f'(x), 0 \rangle$ and $\vec{r}''(x)=\langle 0, f''(x), 0 \rangle$. Therefore

$$\vec{r}'(x) \times \vec{r}''(x) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x) \hat{k}$$

$$|\vec{r}'(x) \times \vec{r}''(x)| = f''(x)$$

$$\text{Now, } |\vec{r}'(x)| = \sqrt{1 + (f'(x))^2}$$

So

$$K(x) = \frac{f''(x)}{(1 + (f'(x))^2)^{3/2}}$$

Therefore $K(x)=0$ at an inflection point because at an inflection point $f''=0$.

2. The statement is not true in general. For example, the line $\vec{r}(t) = \langle t, t, t \rangle$ can be written also as $\vec{w}(s) = \langle 2s, 2s, 2s \rangle$. Then $\vec{r}'(t) = \langle 1, 1, 1 \rangle$ and $\vec{w}'(s) = \langle 2, 2, 2 \rangle$. Since $\vec{r}'(t) \neq \vec{w}'(s)$ it is not true that different parametrizations of the same curve result in identical tangent vectors at a given point on the curve.

3. The length of the curve described by $\vec{r}(t) = \langle 2t^{3/2}, \cos(2t), \sin(2t) \rangle$ $0 \leq t \leq 1$ is given by

$$s = \int_0^1 |\vec{r}'(t)| dt$$

$$\vec{r}'(t) = \left\langle 2\left(\frac{3}{2}\right)t^{1/2}, -2\sin(2t), 2\cos(2t) \right\rangle$$

$$|\vec{r}'(t)| = \sqrt{9t + 4\sin^2(2t) + 4\cos^2(2t)}$$

$$= \sqrt{9t + 4}$$

$$s = \int_0^1 \sqrt{9t+4} dt, \quad \begin{array}{l} \text{if } 9t+4 = u \\ 9dt = du \end{array}$$

$$\begin{aligned}
 s &= \frac{1}{9} \int_4^{13} u^{1/2} du = \frac{1}{9} \left(\frac{2}{3} \right) u^{3/2} \Big|_4^{13} \\
 &= \frac{2}{27} \left[13^{3/2} - 4^{3/2} \right] = \frac{2}{27} (13^{3/2} - 8)
 \end{aligned}$$

4. Given $x = 3\cos t$ and $y = 4\sin t$ we can write $\vec{r}(t) = \langle 3\cos t, 4\sin t, 0 \rangle$.

$$\vec{r}'(t) = \langle -3\sin t, 4\cos t, 0 \rangle$$

$$\vec{r}''(t) = \langle -3\cos t, -4\sin t, 0 \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \uparrow & \uparrow & \uparrow \\ -3\sin t & 4\cos t & 0 \\ -3\cos t & -4\sin t & 0 \end{vmatrix}$$

$$\begin{aligned}
 &= (12\sin^2 t + 12\cos^2 t) \hat{k} \\
 &= 12 \hat{k}
 \end{aligned}$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = 12$$

$$\begin{aligned}
 |\vec{r}'(t)| &= \sqrt{9\sin^2 t + 16\cos^2 t} \\
 &= \sqrt{9 + 7\cos^2 t}
 \end{aligned}$$

$$K(t) = \frac{12}{(9 + 7\cos^2 t)^{3/2}}$$

$$@ (3, 0) \quad t = 0 \quad @ (0, 4) \quad t = \frac{\pi}{2}$$

Thus

$$K(0) = \frac{12}{(9+7)^{3/2}} = \frac{12}{(16)^{3/2}} = \frac{12}{64} = \frac{3}{16}$$

$$K\left(\frac{\pi}{2}\right) = \frac{12}{(9)^{3/2}} = \frac{12}{27} = \frac{4}{9}$$

$$5. \text{ Since } \vec{a} = 6t\hat{i} + 12t^2\hat{j} - 6t\hat{k},$$

$$\vec{v} = \int \vec{a} \, dt = 6 \frac{t^2}{2} \hat{i} + 12 \frac{t^3}{3} \hat{j} - 6 \frac{t^2}{2} \hat{k} + \vec{C}$$

$$= 3t^2\hat{i} + 4t^3\hat{j} - 3t^2\hat{k} + \vec{C}$$

$$\text{Since } \vec{v}_0 = \hat{i} - \hat{j} + 3\hat{k} \text{ we have } \vec{C} = \vec{v}_0$$

$$\text{So } \vec{v} = (3t^2 + 1)\hat{i} + (4t^3 - 1)\hat{j} - (3t^2 - 3)\hat{k}$$

$$\vec{r} = \int \vec{v} dt = \left(3 \frac{t^3}{3} + t\right) \hat{i} + \left(4 \frac{t^4}{4} - t\right) \hat{j} - \left(3 \frac{t^3}{3} - 3t\right) \hat{k} + \vec{D}$$

Since $\vec{r}_0 = \vec{0}$ $\vec{D} = \vec{0}$

$$\begin{aligned} \text{So } \vec{r} &= (t^3 + t) \hat{i} + (t^4 - t) \hat{j} - (t^3 - 3t) \hat{k} \\ &= (t^3 + t) \hat{i} + (t^4 - t) \hat{j} + (3t - t^3) \hat{k} \end{aligned}$$

Ex - $\vec{r}(t) = \left\langle \int_0^t \sin\left(\frac{1}{2}\pi\theta^2\right) d\theta, \int_0^t \cos\left(\frac{1}{2}\pi\theta^2\right) d\theta, 0 \right\rangle$

Using the fundamental theorem of calculus:

$$\vec{r}'(t) = \left\langle \sin\left(\frac{1}{2}\pi t^2\right), \cos\left(\frac{1}{2}\pi t^2\right), 0 \right\rangle$$

$$\vec{r}''(t) = \left\langle \pi t \cos\left(\frac{1}{2}\pi t^2\right), -\pi t \sin\left(\frac{1}{2}\pi t^2\right), 0 \right\rangle$$

$$|\vec{r}'(t)| = 1$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin\left(\frac{1}{2}\pi t^2\right) & \cos\left(\frac{1}{2}\pi t^2\right) & 0 \\ \pi t \cos\left(\frac{1}{2}\pi t^2\right) & -\pi t \sin\left(\frac{1}{2}\pi t^2\right) & 0 \end{vmatrix}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \left[-\pi t \sin^2\left(\frac{1}{2}\pi t^2\right) - \pi t \cos^2\left(\frac{1}{2}\pi t^2\right) \right] \hat{k}$$

$$= -\pi t \hat{k}$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{\pi^2 t^2} = \pi |t|$$

$$\therefore \underline{\kappa(t) = \pi |t|}$$

$$7. \vec{r}(t) = \left\langle vt \cos \alpha, vt \sin \alpha - \frac{g}{2} t^2 \right\rangle$$

$$\vec{r}'(t) = \langle v \cos \alpha, v \sin \alpha - gt \rangle$$

$$|\vec{r}'(t)| = \sqrt{v^2 \cos^2 \alpha + (v \sin \alpha - gt)^2}$$

$$= \sqrt{v^2 \cos^2 \alpha + v^2 \sin^2 \alpha - 2vg t \sin \alpha + g^2 t^2}$$

$$= \sqrt{v^2 - 2vg t \sin \alpha + g^2 t^2}$$

Factorizing g^2 :

$$|\vec{r}'(t)| = g \sqrt{\frac{v^2}{g^2} - \frac{2v \sin \alpha t}{g} + t^2}$$

Completing squares:

$$|\vec{r}'(t)| = g \sqrt{\frac{v^2}{g^2} - \frac{v^2 \sin^2 \alpha}{g^2} + \left(t - \frac{v}{g} \sin \alpha\right)^2}$$
$$= g \sqrt{\frac{v^2}{g^2} \cos^2 \alpha + \left(t - \frac{v}{g} \sin \alpha\right)^2}$$

Using:

$$\int \sqrt{a^2 + u^2} du = \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln(u + \sqrt{a^2 + u^2}) + C$$

$$L = \int_0^{t_{\text{land}}} |\vec{r}'| dt = \int_0^{t_{\text{land}}} g \sqrt{\frac{v^2}{g^2} \cos^2 \alpha + \left(t - \frac{v}{g} \sin \alpha\right)^2} dt$$

t_{land} is equal to the time the projectile takes to land. Therefore $t_{\text{land}} = \frac{2v}{g} \sin \alpha$.

$$L = g \int_0^{\frac{2v}{g} \sin \alpha} \sqrt{\frac{v^2}{g^2} \cos^2 \alpha + \left(t - \frac{v}{g} \sin \alpha\right)^2} dt$$

$$L = g \left[\frac{\left(t - \frac{v}{g} \sin \alpha\right)^2}{2} + \sqrt{\frac{v^2}{g^2} \cos^2 \alpha + \left(t - \frac{v}{g} \sin \alpha\right)^2} \right. \\ \left. + \frac{v^2}{2g^2} \cos^2 \alpha \ln \left(t - \frac{v}{g} \sin \alpha + \sqrt{\frac{v^2}{g^2} \cos^2 \alpha + \left(t - \frac{v}{g} \sin \alpha\right)^2} \right) \right]$$

evaluated from 0 to $\frac{2v}{g} \sin \alpha$. After some simplification:

$$L = \frac{g}{2} \left[\frac{v^2 \sin \alpha}{g^2} + \frac{v^2 \cos^2 \alpha}{g^2} \ln \left(\frac{v(\sin \alpha + 1)}{g} \right) \right] \\ - \frac{g}{2} \left[-\frac{v^2 \sin \alpha}{g^2} + \frac{v^2 \cos^2 \alpha}{g^2} \ln \left(\frac{v(1 - \sin \alpha)}{g} \right) \right]$$

Simplifying further:

$$L = \frac{v^2 \sin \alpha}{g} + \frac{v^2 \cos^2 \alpha}{2g} \ln \left(\frac{1 + \sin \alpha}{1 - \sin \alpha} \right)$$

Since we want to maximize L , we differentiate it with respect to α :

$$L' = \frac{v^2}{g} \cos \alpha + \frac{v^2}{2g} \left[-2 \cos \alpha \sin \alpha \ln \left(\frac{1 + \sin \alpha}{1 - \sin \alpha} \right) + \right. \\ \left. \cos^2 \alpha \left(\frac{(1 - \sin \alpha)}{(1 + \sin \alpha)} \cdot \frac{(1 - \sin \alpha) \cos \alpha - (1 + \sin \alpha)(-\cos \alpha)}{(1 - \sin \alpha)^2} \right) \right]$$

Simplifying:

$$L' = \frac{v^2 \cos \alpha}{g} \left(2 - \sin \alpha \ln \left(\frac{1 + \sin \alpha}{1 - \sin \alpha} \right) \right)$$

L has critical points for $0 < \alpha < \frac{\pi}{2}$
when $L' = 0$

$\therefore \cos \alpha \neq 0$ and

$$2 - \sin \alpha \ln \left(\frac{1 + \sin \alpha}{1 - \sin \alpha} \right) = 0$$

Using maple to find the value of α that satisfies the equation, we find

$$\alpha \approx 0.9855 \text{ rad}$$

or

$$\alpha \approx 56.43^\circ$$
