Name: Marco A. Montes de Oca — MATH 243 - Quiz 1

Section: 50 and 51 — February 15-16, 2012

Please SHOW ALL WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Suppose $\vec{r}_1 = 2\hat{\imath} - \hat{\jmath}$, $\vec{r}_2 = \hat{\imath} - 3\hat{\jmath}$. Write $\vec{r}_3 = 5\hat{\imath} + 15\hat{\jmath}$ as a linear combination of $\vec{r}_1$ and $\vec{r}_2$; that is, find scalars a, b such that $\vec{r}_3 = a\vec{r}_1 + b\vec{r}_2$.

   We want to find the values of a and b such that $\vec{r}_3 = a\vec{r}_1 + b\vec{r}_2$. Since we know that $\vec{r}_3 = 5\hat{\imath} + 15\hat{\jmath}$, we can set $a\vec{r}_1 + b\vec{r}_2 = 5\hat{\imath} + 15\hat{\jmath}$, which means that $a(2\hat{\imath} - \hat{\jmath}) + b(\hat{\imath} - 3\hat{\jmath}) = 5\hat{\imath} + 15\hat{\jmath}$.

   Multiplying and rearranging terms we get $2a\hat{\imath} - a\hat{\jmath} + b\hat{\imath} - 3b\hat{\jmath} = (2a+b)\hat{\imath} + (-a-3b)\hat{\jmath} = 5\hat{\imath} + 15\hat{\jmath}$. This means that $2a + b = 5$ (1) and $-a - 3b = 15$ (2). Multiplying (2) by 2 and adding it to (1) we get $2a + b + (-2a - 6b) = 5 + 30$ which can simplified to $b - 6b = -5b = 35$ which means that $b = \frac{35}{-5} = -7$. To find the value of a we substitute b in (2), for example, and get $-a - 3(-7) = -a + 21 = 15$ which means that $a = 21 - 15 = 6$.

   Thus, $\vec{r}_3$ can be written as $\vec{r}_3 = 6\vec{r}_1 - 7\vec{r}_2$.

2. (25 pts) The position vectors of points P and Q are given by $\vec{r}_1 = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $\vec{r}_2 = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$ respectively. Find the vector $\vec{PQ}$ in terms of $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, and find its magnitude.

   The position vectors of points P and Q can be thought of as the vectors that start at the origin and end at the points P and Q. Thus, the vector $\vec{PQ}$ is the vector that would "move us" from P to Q. This vector can be found by subtracting $\vec{r}_1$ from $\vec{r}_2$; that is, $\vec{PQ} = \vec{r}_2 - \vec{r}_1 = 4\hat{\imath} - 3\hat{\jmath} + 2\hat{k} + (-1)(2\hat{\imath} + 3\hat{\jmath} - \hat{k}) = (4-2)\hat{\imath} + (-3-3)\hat{\jmath} + (2-(-1))\hat{k} = 2\hat{\imath} - 6\hat{\jmath} + 3\hat{k}$.

   The magnitude of the vector $\vec{PQ} = 2\hat{\imath} - 6\hat{\jmath} + 3\hat{k}$ is given by $|\vec{PQ}| = \sqrt{2^2 + (-6)^2 + 3^2} = \sqrt{4 + 36 + 9} = \sqrt{49} = 7$.

3. (25 pts) Determine the value of α so that $\vec{A} = \langle 2, \alpha, 1\rangle$ and $\vec{B} = \langle 1, 3, -8\rangle$ are perpendicular.

   If $\vec{A}$ and $\vec{B}$ are perpendicular, then $\vec{A} \cdot \vec{B}$ must be equal to zero. Thus, we have that

   $\vec{A} \cdot \vec{B} = 2(1) + \alpha(3) + (-8)(1) = 2 + 3\alpha - 8 = 3\alpha - 6 = 0$. This means that $3\alpha = 6$, which in turn means that $\alpha = \frac{6}{3} = 2$.

4. (25 pts) Find the acute angle formed by two diagonals of a cube.

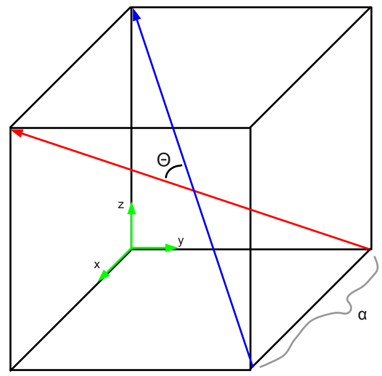

Figure 1: Cube and its diagonals as vectors. The origin of the coordinate system is marked by three green arrows.

The problem is to find the angle θ formed by the vectors aligned along the diagonals of a cube (for an example, see Fig. 1).

The components of the diagonal vectors depend on the length of the cube's sides. Let's assume that these sides are of length α. Based on the figure, the components of the blue vector, which we will call $\vec{b}$, are $\langle -\alpha, -\alpha, \alpha \rangle$. The components of the red vector, which we will call $\vec{r}$, are $\langle \alpha, -\alpha, \alpha \rangle$.

Let's use the dot product to find $\cos(\theta)$. In particular, remember that $\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$ for some vectors $\vec{a}$ and $\vec{b}$. We have

$$|\vec{r}| = \sqrt{(\alpha)^2 + (-\alpha)^2 + (\alpha)^2} = \sqrt{\alpha^2 + \alpha^2 + \alpha^2 +} = \sqrt{3\alpha^2} = \alpha\sqrt{3}.$$
$$|\vec{b}| = \sqrt{(-\alpha)^2 + (-\alpha)^2 + (\alpha)^2} = \sqrt{\alpha^2 + \alpha^2 + \alpha^2 +} = \sqrt{3\alpha^2} = \alpha\sqrt{3}.$$
$$\vec{r} \cdot \vec{b} = (\alpha)(-\alpha) + (-\alpha)(-\alpha) + (\alpha)(\alpha) = -\alpha^2 + \alpha^2 + \alpha^2 = \alpha^2.$$

Thus, $\cos(\theta) = \frac{\vec{r} \cdot \vec{b}}{|\vec{r}||\vec{b}|} = \frac{\alpha^2}{(\alpha\sqrt{3})(\alpha\sqrt{3})} = \frac{\alpha^2}{3\alpha^2} = \frac{1}{3}$.

This means that $\theta = \arccos(\frac{1}{3})$.

Note that many other definitions of the diagonal vectors are possible. Some of those combinations could give you the obtuse angle instead of the acute angle (which is the one we are interested in). For example, if we had used $\vec{r} = \langle -\alpha, \alpha, -\alpha \rangle$ (the negative of the version we actually used), then the angle between the vectors $\vec{r}$ and $\vec{b}$ would be greater than $\pi/2$ and the result would have been $\theta = \arccos(-\frac{1}{3})$ (Check it yourself!), which is an obtuse angle. Even if you do this, the acute angle is always the same: $\arccos(\frac{1}{3})$.

Bonus (10 pts): If you are given the coordinates of the three vertices of a triangle PQR, how would you use the dot product to determine whether PQR is a right triangle?

The idea is to find the vectors $\vec{PQ}$, $\vec{PR}$ and $\vec{QR}$. Then if the dot product of any of these vectors is zero, then we will be sure that the angle between them is $\pi/2$, that is, that they are perpendicular to each other, and therefore, that the triangle PQR is a right triangle.