Name: Marco A. Montes de Oca — MATH 243 - Quiz 2

Section: 50 and 51 — February 29, March 1, 2012

Please SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Show that $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$.

   Solution: Considering that $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin(\theta)$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$, then $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = (|\vec{a}||\vec{b}|\sin(\theta))^2 + ||\vec{a}||\vec{b}|\cos(\theta)|^2 = |\vec{a}|^2|\vec{b}|^2\sin^2(\theta) + |\vec{a}|^2|\vec{b}|^2\cos^2(\theta) = |\vec{a}|^2|\vec{b}|^2(\sin^2(\theta) + \cos^2(\theta)) = |\vec{a}|^2|\vec{b}|^2$.

2. (25 pts) Consider two lines in 3D space. One of them, L_1, is defined by the vector function $\vec{r}_1(t) = \langle 1, 2, 0\rangle + t\langle 1, -1, 2\rangle$. The other line, L_2, passes through the point $P(a, 0, 0)$ and its direction vector is $\langle -3, 1, 1\rangle$. What should the value of a be to make sure that L_1 and L_2 intersect? What would be the intersection point?

   Solution: The vector equations of the lines are:

   L_1: $\vec{r}_1(t) = \langle 1, 2, 0\rangle + t\langle 1, -1, 2\rangle = \langle 1 + t, 2 - t, 2t\rangle$
   L_2: $\vec{r}_2(s) = \langle a, 0, 0\rangle + s\langle -3, 1, 1\rangle = \langle a - 3s, s, s\rangle$

   Therefore, the parametric equations of the lines are:

   L_1: $x = 1 + t$, $y = 2 - t$, $z = 2t$
   L_2: $x = a - 3s$, $y = s$, $z = s$

   If the two lines intersect at a point, then

   $$\begin{aligned} (1) \quad 1 + t &= a - 3s \\ (2) \quad 2 - t &= s \\ (3) \quad 2t &= s \end{aligned}$$

   Substituting s from (2) in (3) we obtain $2t = 2 - t$, which means that $2t + t = 3t = 2$ and therefore $t = \frac{2}{3}$. Substituting t back in (2) we get $2 - (\frac{2}{3}) = \frac{6}{3} - \frac{2}{3} = \frac{4}{3} = s$. We can now find the value of a that would allow the two lines intersect. We do this by substituting the values of t and s in (1):

   $1 + \frac{2}{3} = a - 3(\frac{4}{3})$. Thus, $a = \frac{5}{3} + \frac{12}{3} = \frac{17}{3}$.

   Now, if $a = \frac{17}{3}$, the point of intersection would be $(\frac{5}{3}, \frac{4}{3}, \frac{4}{3})$.

3. (25 pts) Find a normal vector to the plane that passes through the point $(1,1,1)$ and contains the line with parametric equations $x = 1 + 3t$, $y = 1 + 2t$, $z = 2 - 3t$.

Solution: In order to find a normal vector to the plane with the characteristics described in the problem's statement, we need to find at least two vectors that are on the plane so that we can use the cross product to find an orthogonal vector to those vectors, and therefore to the plane.

We can find two vectors by "linking" the given point to two other points on the line described by the parametric equations. Let us find two such points. One point can be found of we let $t = 0$. In this case, we will find that the point P_1 has coordinates $(1,1,2)$. Similarly, if we let $t = 1$, we will obtain the point P_2 with coordinates $(4,3,-1)$.

We can now find two vectors on the plane. These vectors are:

$\vec{v}_1 = \langle 1-1, 1-1, 2-1 \rangle = \langle 0,0,1 \rangle$
$\vec{v}_2 = \langle 4-1, 3-1, -1-1 \rangle = \langle 3,2,-2 \rangle$

Then, a normal vector to the plane is

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ 3 & 2 & -2 \end{vmatrix} = -2\hat{i} + 3\hat{j}$$

The negative of $\vec{n}$, that is $-\vec{n}$, is also a normal vector to the plane described in the problem's statement.

4. (25 pts) Describe with words the surface $y^2 + 2z^2 - 6y - x + 10 = 0$. You may use a sketch to help you describe the surface but it is not obligatory.

Solution: Let us rewrite the equation in standard form so that the identification process becomes easier.

$y^2 + 2z^2 - 6y - x + 10 = 0$
$(y^2 - 6y) + 2z^2 = x - 10$

Completing the square for y:

$(y^2 - 6y + 9) + 2z^2 = x - 10 + 9$
$(y-3)^2 + 2z^2 = x - 1$

Dividing by 2:

$\frac{(y-3)^2}{2} + z^2 = \frac{x-1}{2}$

This equation represents an elliptic paraboloid parallel to the x-axis. The elliptic traces on the plane $x = k$ for $x > 1$ are ellipses whose major axes are parallel to the y-axis. The vertex of the paraboloid is located at $(1,3,0)$. See Figure 1 for a 3D representation of this paraboloid.

Bonus (10 pts): Is the vector $\langle -3,-2,-6 \rangle$ orthogonal to the plane $3x + 2y + 6z = 6$? Explain why you think it is or it is not.

Yes, the vector $\vec{v} = \langle -3,-2,-6 \rangle$ is orthogonal to the plane $3x + 2y + 6z = 6$. The reason is that $\vec{v}$ is simply the negative of the normal vector $\vec{n} = \langle 3,2,6 \rangle$ of the plane. Thus, both vectors are orthogonal to the plane $3x+2y+6z = 6$.

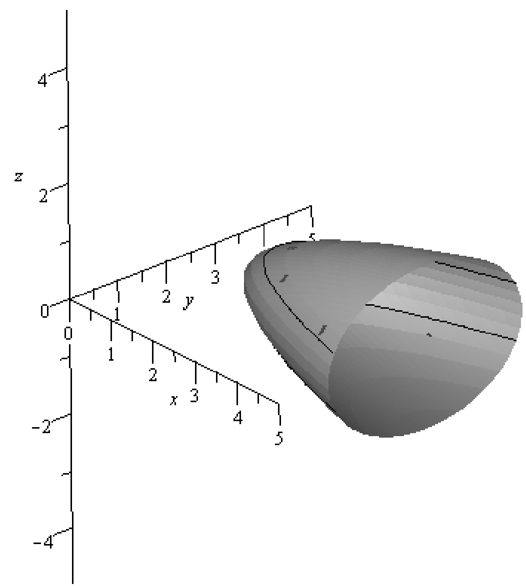

Figure 1: Elliptic paraboloid