Name: Marco A. Montes de Oca — MATH 243 - Quiz 3

Section: 50 and 50 — March 14 & 15, 2012

Please SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Suppose $\vec{A}$, $\vec{B}$, $\vec{C}$ are differentiable functions of a scalar u. Find $\frac{\mathrm{d}}{\mathrm{d}u}\left(\vec{A}\cdot(\vec{B}\times\vec{C})\right)$

   Solution: $\frac{\mathrm{d}}{\mathrm{d}u}\left(\vec{A}\cdot(\vec{B}\times\vec{C})\right) = \frac{\mathrm{d}\vec{A}}{\mathrm{d}u}\cdot(\vec{B}\times\vec{C}) + \vec{A}\cdot\frac{\mathrm{d}}{\mathrm{d}u}\left(\vec{B}\times\vec{C}\right) =$
   $\frac{\mathrm{d}\vec{A}}{\mathrm{d}u}\cdot(\vec{B}\times\vec{C}) + \vec{A}\cdot\left(\frac{\mathrm{d}\vec{B}}{\mathrm{d}u}\times\vec{C} + \vec{B}\times\frac{\mathrm{d}\vec{C}}{\mathrm{d}u}\right) =$
   $\frac{\mathrm{d}\vec{A}}{\mathrm{d}u}\cdot(\vec{B}\times\vec{C}) + \vec{A}\cdot\left(\frac{\mathrm{d}\vec{B}}{\mathrm{d}u}\times\vec{C}\right) + \vec{A}\cdot\left(\vec{B}\times\frac{\mathrm{d}\vec{C}}{\mathrm{d}u}\right)$

2. (25 pts) Suppose $\vec{A}$ is a differentiable function of a scalar u. Show that $\vec{A}\cdot\frac{\mathrm{d}\vec{A}}{\mathrm{d}u} = A\frac{\mathrm{d}A}{\mathrm{d}u}$, where $A = |\vec{A}|$.

   Solution: If $\vec{A} = \langle A_1, A_2, A_3\rangle$, then $A = \sqrt{A_1^2 + A_2^2 + A_3^2}$. Therefore,
   $\frac{\mathrm{d}A}{\mathrm{d}u} = \frac{1}{2}\left(A_1^2 + A_2^2 + A_3^2\right)^{-1/2}\left(2A_1\frac{\mathrm{d}A_1}{\mathrm{d}u} + 2A_2\frac{\mathrm{d}A_2}{\mathrm{d}u} + 2A_3\frac{\mathrm{d}A_3}{\mathrm{d}u}\right)$.
   $\frac{\mathrm{d}A}{\mathrm{d}u} = \frac{1}{2}\left(A_1^2 + A_2^2 + A_3^2\right)^{-1/2} 2\left(A_1\frac{\mathrm{d}A_1}{\mathrm{d}u} + A_2\frac{\mathrm{d}A_2}{\mathrm{d}u} + A_3\frac{\mathrm{d}A_3}{\mathrm{d}u}\right)$.
   $\frac{\mathrm{d}A}{\mathrm{d}u} = \frac{\left(A_1\frac{\mathrm{d}A_1}{\mathrm{d}u} + A_2\frac{\mathrm{d}A_2}{\mathrm{d}u} + A_3\frac{\mathrm{d}A_3}{\mathrm{d}u}\right)}{\left(A_1^2 + A_2^2 + A_3^2\right)^{1/2}}$.
   $\left(A_1^2 + A_2^2 + A_3^2\right)^{1/2}\frac{\mathrm{d}A}{\mathrm{d}u} = \left(A_1\frac{\mathrm{d}A_1}{\mathrm{d}u} + A_2\frac{\mathrm{d}A_2}{\mathrm{d}u} + A_3\frac{\mathrm{d}A_3}{\mathrm{d}u}\right)$.
   $A\frac{\mathrm{d}A}{\mathrm{d}u} = \left(A_1\frac{\mathrm{d}A_1}{\mathrm{d}u} + A_2\frac{\mathrm{d}A_2}{\mathrm{d}u} + A_3\frac{\mathrm{d}A_3}{\mathrm{d}u}\right) = \langle A_1, A_2, A_3\rangle\cdot\langle\frac{\mathrm{d}A_1}{\mathrm{d}u}, \frac{\mathrm{d}A_2}{\mathrm{d}u}, \frac{\mathrm{d}A_2}{\mathrm{d}u}\rangle$.
   $A\frac{\mathrm{d}A}{\mathrm{d}u} = \vec{A}\cdot\frac{\mathrm{d}\vec{A}}{\mathrm{d}u}$.

3. (25 pts) Use your favorite method to find the curvature of $\vec{r}(x) = \langle x, 2x^3 - 4x^2 + x - 2\rangle$ at $x = 2/3$.

   Solution: Let us use the expression $\kappa(x) = \frac{|\vec{r}\,'(x)\times\vec{r}\,''(x)|}{|\vec{r}\,'(x)|^3}$.

   $\vec{r}\,'(x) = \langle 1, 6x^2 - 8x + 1\rangle$, and $\vec{r}\,''(x) =< 0, 12x - 8 >$. At $x = 2/3$, these vectors are $\vec{r}\,'(\frac{2}{3}) = \langle 1, 6(\frac{2}{3})^2 - 8(\frac{2}{3}) + 1\rangle$ and $\vec{r}\,''(\frac{2}{3}) = \langle 0, 12(\frac{2}{3}) - 8\rangle =< 0, 0 >$. Thus, extending the vectors $\vec{r}\,'(x)$ and $\vec{r}\,''(x)$ to three dimensions, we find that $|\vec{r}\,'(\frac{2}{3})\times\vec{r}\,''(\frac{2}{3})| = 0$ and therefore $\kappa = 0$ at $x = 2/3$.

4. (25 pts) Reparametrize the curve $\vec{r}(t) = e^t\hat{\imath} + e^t\sin(t)\hat{\jmath} + e^t\cos(t)\hat{k}$ with respect to the arc length measured from the point $(1, 0, 1)$ in the direction of increasing t.

   Solution: We need to evaluate the integral $s(t) = \int_a^t |\vec{r}\,'(u)|\,du$, find a function $t(s)$ and reparametrize the original vector function.

Since the initial point is $(1, 0, 1)$, then $a = 0$. Thus, $s(t) = \int_0^t |\langle e^u, e^u(\sin(u) + \cos(u)), e^u(\cos(u) - \sin(u))\rangle|\, du$.
$s(t) = \int_0^t \sqrt{(e^t)^2 + (e^t)^2(\sin(u) + \cos(u))^2 + (e^t)^2(\cos(u) - \sin(u))^2}\, du$.
$s(t) = \int_0^t e^t \sqrt{1 + (\sin(u) + \cos(u))^2 + (\cos(u) - \sin(u))^2}\, du$.
$s(t) = \int_0^t e^t \sqrt{1 + \sin^2(u) + 2\sin(u)\cos(u) + \cos^2(u) + \cos^2(u) - 2\cos(u)\sin(u) + \sin^2(u)}\, du$.
$s(t) = \int_0^t e^t \sqrt{3}\, du$.
$s(t) = \sqrt{3}\, [e^t]_0^t$.
$s(t) = \sqrt{3}(e^t - 1)$.
$t(s) = \ln\left(\frac{s}{\sqrt{3}} + 1\right)$.

Reparametrizing:
$\vec{r}(t(s)) = \langle e^{\ln\left(\frac{s}{\sqrt{3}}+1\right)}, e^{\ln\left(\frac{s}{\sqrt{3}}+1\right)} \sin(\ln\left(\frac{s}{\sqrt{3}} + 1\right)), e^{\ln\left(\frac{s}{\sqrt{3}}+1\right)} \cos(\ln\left(\frac{s}{\sqrt{3}} + 1\right))\rangle =$
$\langle \frac{s}{\sqrt{3}} + 1, \left(\frac{s}{\sqrt{3}} + 1\right) \sin(\ln\left(\frac{s}{\sqrt{3}} + 1\right)), \left(\frac{s}{\sqrt{3}} + 1\right) \cos(\ln\left(\frac{s}{\sqrt{3}} + 1\right))\rangle$.

Bonus (10 pts) If the norm of a vector function $\vec{r}(t)$ is constant, what is the relationship between $\vec{r}(t)$ and $\vec{r}\,'(t)$?

Solution: They are perpendicular, that is, $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$.