Name: Marco A. Montes de Oca — MATH 243 - Quiz 4

Section: 50 & 51 — April 4 & 5, 2012

Please SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Find the first partial derivatives of $T(p,q,r) = p\ln(q+e^r)$

$\frac{\partial T}{\partial p} = \ln(p+e^r)\frac{\partial p}{\partial p} = \ln(p+e^r)$
$\frac{\partial T}{\partial q} = p\frac{\partial \ln(q+e^r)}{\partial q} = p\frac{1}{q+e^r}(1) = \frac{p}{q+e^r}$
$\frac{\partial T}{\partial r} = p\frac{\partial \ln(q+e^r)}{\partial r} = p\frac{1}{q+e^r}(e^r) = \frac{pe^r}{q+e^r}$

2. (25 pts) Find an equation of the tangent plane to the surface $z = e^x\cos(y)$ at the point $(0,0,1)$.

An equation of the tangent plane to the surface $z = f(x,y)$ at the point $P(x_0,y_0,z_0)$ is
$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$. Then, we need to find the first partial derivatives of $f(x,y) = e^x\cos(y)$.

$f_x(x,y) = e^x\cos(y)$, then $f_x(0,0) = 1$
$f_y(x,y) = -e^x\sin(y)$, then $f_y(0,0) = 0$.

An equation of the tangent plane to the surface $z = e^x\cos(y)$ at the point $(0,0,1)$ is then
$z - 1 = 1(x-0) + 0(y-0) = x$ or $z = x+1$.

3. (25 pts) If $u = x^2y^3 + z^4$, where $x = p+3p^2$, $y = pe^p$, and $z = p\sin(p)$, use the Chain Rule to find du/dp.

$\frac{du}{dp} = \frac{\partial u}{\partial x}\frac{dx}{dp} + \frac{\partial u}{\partial y}\frac{dy}{dp} + \frac{\partial u}{\partial z}\frac{dz}{dp}$

$\frac{\partial u}{\partial x} = 2xy^3$, $\frac{\partial u}{\partial y} = 3x^2y^2$, $\frac{\partial u}{\partial z} = 4z^3$.

$\frac{dx}{dp} = 1+6p$, $\frac{dy}{dp} = e^p(1+p)$, $\frac{dz}{dp} = \sin(p) + p\cos(p)$.

$\frac{du}{dp} = 2xy^3(1+6p) + 3x^2y^2(e^p(1+p)) + 4z^3(\sin(p)+p\cos(p))$

4. (25 pts) Find the local maximum and minimum values and saddle point(s) of the function $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2$.

Critical points:
$\frac{\partial f}{\partial x} = -2 - 2x = 0 \rightarrow x = -1$
$\frac{\partial f}{\partial y} = 4 - 8y = 0 \rightarrow y = 1/2$

The point $(-1, 1/2)$ is the only critical point.

The Hessian of f is

$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -8 \end{bmatrix}$ and its determinant is equal to $(-2)(-8) - (0)(0) = 16 > 0$. Therefore, at $(-1, 1/2)$ the function has a local maximum.

Bonus (10 pts): If $\vec{r} = \langle x, y, z \rangle$ and $f = \ln |\vec{r}|$, show that $\nabla f = \dfrac{\vec{r}}{|\vec{r}|^2}$.

$f = \ln(\sqrt{x^2 + y^2 + z^2}) = \ln((x^2 + y^2 + z^2)^{1/2}) = \frac{1}{2} \ln(x^2 + y^2 + z^2)$

$\nabla f = \frac{1}{2} \langle \frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2} \rangle$

$\nabla f = \langle \frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2} \rangle = \frac{\vec{r}}{|\vec{r}|^2}$