Name: Marco A. Montes de Oca — MATH 243 - Quiz 4

Section: 50 & 51 — April 18 & 19, 2012

Please SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) Use Lagrange multipliers to find the absolute maximum and minimum values of $f(x, y) = x^2 - y$ subject to the constraint $x^2 + y^2 = 1$.

   Solution: The Lagrange multiplier equation is:

   $\nabla f(x, y) = \lambda \nabla g(x, y)$, where $g(x, y) = x^2 + y^2 - 1$
   $\langle 2x, -1\rangle = \lambda\langle 2x, 2y\rangle$

   This creates the following system of equations:
   $2x = \lambda 2x$
   $-1 = \lambda 2y$
   $x^2 + y^2 = 1$

   From the first equation, $x(1 - \lambda) = 0$, which means that $x = 0$ and/or $\lambda = 1$. Substituting $x = 0$ in the third equation, means that $y^2 = 1$, or $y = \pm 1$. If $\lambda = 1$, then from the second equation, $y = -\frac{1}{2}$. Then, from the third equation, $x^2 + \frac{1}{4} = 1$, $x^2 = \frac{3}{4}$, which means that $x = \pm\frac{\sqrt{3}}{2}$. We can conclude then that the points $(0, 1)$, $(0, -1)$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{-\sqrt{3}}{2}, -\frac{1}{2})$ are candidate solutions.

   Evaluating the function at the candidate solutions:
   $f(0, 1) = -1$
   $f(0, -1) = 1$
   $f(\frac{\sqrt{3}}{2}, -\frac{1}{2}) = f(-\frac{\sqrt{3}}{2}, -\frac{1}{2}) = \frac{5}{4}$. Therefore, the maximum value is $\frac{5}{4}$ at $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{-\sqrt{3}}{2}, -\frac{1}{2})$ and the minimum is -1 at $(0, 1)$.

2. (25 pts) Evaluate the double integral $\displaystyle\iint_R \frac{xy^2}{x^2+1}\mathrm{d}A$, where $R = \{(x, y) \mid 0 \le x \le 1, -3 \le y \le 3\}$

   Solution: $\iint_R \frac{xy^2}{x^2+1}\mathrm{d}A = \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\mathrm{d}y\mathrm{d}x = \int_0^1 \frac{x}{x^2+1}\left[\frac{y^3}{3}\right]_{-3}^3 \mathrm{d}x = \int_0^1 \frac{x}{x^2+1}\left[\frac{27}{3} - \left(-\frac{27}{3}\right)\right]\mathrm{d}x = 18\int_0^1 \frac{x}{x^2+1}\mathrm{d}x$.

   Using the substitution $u = x^2 + 1$, we get that $18\int_0^1 \frac{x}{x^2+1}\mathrm{d}x = 18\frac{1}{2}\int_1^2 \frac{\mathrm{d}u}{u} = 9\left[\ln(u)\right]_1^2 = 9\ln(2)$.

3. (25 pts) Evaluate the double integral $\displaystyle\iint_D x\sin(y)\mathrm{d}A$, where D is enclosed by the curves $y = 0$, $y = x^2$, and $x = 1$.

   Solution: $\iint_D x\sin(y)\mathrm{d}A = \int_0^1 \int_0^{x^2} x\sin(y)\mathrm{d}y\mathrm{d}x = \int_0^1 x\left[-\cos(y)\right]_0^{x^2}\mathrm{d}x = \int_0^1 x\left[-\cos(x^2) - (-\cos(0))\right]\mathrm{d}x =$
   $\int_0^1 x\left[-\cos(x^2) + 1\right]\mathrm{d}x = -\int_0^1 x\cos(x^2)\mathrm{d}x + \int_0^1 x\mathrm{d}x$.

Using the substitution $u = x^2$, we see that $-\int_0^1 x\cos(x^2)\mathrm{d}x + \int_0^1 x\mathrm{d}x = -\frac{1}{2}\int_0^1 \cos(u)\mathrm{d}u + [\frac{x^2}{2}]_0^1 = -\frac{1}{2}[\sin(u)]_0^1 + \frac{1}{2} = -\frac{1}{2}\sin(1) + \frac{1}{2} = \frac{1}{2}(1 - \sin(1))$.

4. (25 pts) Find the Jacobian of the transformation $x = e^{-r}\sin(\theta)$, $y = e^r\cos(\theta)$.

Solution:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} -e^{-r}\sin(\theta) & e^{-r}\cos(\theta) \\ e^r\cos(\theta) & -e^r\sin(\theta) \end{vmatrix} = \sin^2(\theta) - \cos^2(\theta)$$

Bonus (10 pts): By making an appropriate change of variables, evaluate the integral $\iint_R e^{x+y}\mathrm{d}A$, where R is given by the inequality $|x| + |y| \leq 1$

Solution: The inequality $|x| + |y| \leq 1$ can be decomposed into four inequalities based on the signs of both x and y. See table below:

$x < 0$?	$y < 0$?	Resulting inequality
No	No	$x + y \leq 1$
No	Yes	$x - y \leq 1$
Yes	No	$-x + y \leq 1$
Yes	Yes	$-x - y \leq 1$

The region that is described by these inequalities is the solid square whose vertices are $(-1,0)$, $(0,1)$, $(1,0)$, and $(0,-1)$.

Considering the function to be integrated, and the inequalities that define R, it is apparent that an appropriate transformation is:

$u = x + y$, and $v = x - y$.

With this transformation, the new region of integration S is a solid square bounded by the equations $u = 1$, $u = -1$, $v = -1$, and $v = 1$. A much simpler region, than the original one.

The Jacobian of the transformation is:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1 - (-1) = -2$$

Therefore $\mathrm{d}u\mathrm{d}v = |-2|\mathrm{d}x\mathrm{d}y$.

Putting all these elements together:

$\iint_R e^{x+y}\mathrm{d}A = \int_{-1}^1\int_{-1}^1 e^u(\frac{1}{2})\mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_{-1}^1\int_{-1}^1 e^u\mathrm{d}u\mathrm{d}v = \frac{1}{2}\int_{-1}^1[e^u]_{-1}^1\mathrm{d}v = \frac{1}{2}\int_{-1}^1[e - e^{-1}]\mathrm{d}v = \frac{1}{2}\left[ev - e^{-1}v\right]_{-1}^1 = \frac{1}{2}\left[e - e^{-1}\right] - \frac{1}{2}\left[-e + e^{-1}\right] = e - e^{-1}$.