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Section: 50 & 51 — May 2 & 3, 2012

Please SHOW ALL YOUR WORK as partial credit may be given; note all relevant equations, ideas, theorems, sketches, etc., to show what you know. Simplify wherever possible to make your answer more compact and neat. (Otherwise, if your answer cannot be simplified then leave it as is.) DO NOT leave your answer as a complex fraction. Answers without justification will be heavily penalized.

1. (25 pts) A particle moves with a velocity completely determined by a velocity field. The trajectory of the particle can be described by the path given by the parametric equations $x(t) = t$, $y(t) = t^2$. Find an expression for the velocity field that moves the particle.

   Solution 1: If a particle is moved by a velocity field, then at each point along its trajectory the vectors of the velocity field are tangent to the particle's trajectory.

   If the particle's trajectory is given by the parametric equations $x(t) = t$, and $y(t) = t^2$, this means that a vector function that describes the particle's trajectory is $\vec{r}(t) = \langle t, t^2 \rangle$. A tangent vector at every point along the trajectory is given by $\vec{r}\,'(t) = \langle 1, 2t \rangle$. Thus, an expression for the velocity field is $\langle 1, 2t \rangle$, which in terms of the original variables is $\vec{v}(x, y) = \langle 1, 2x \rangle$.

   Solution 2: Another way to solve this problem is to remember that if a vector function represents the position of a particle at any time, then its derivative is its velocity. We are given the particle's position function $\vec{r}(t) = \langle t, t^2 \rangle$. The particle's velocity is then $\vec{r}\,'(t) = \langle 1, 2t \rangle$. The particle's velocity is equal to the velocity field the particle is moving in along the particle's trajectory. Thus, $\vec{v}(x, y) = \langle 1, 2x \rangle$.

2. (25 pts) Find the work done in moving a particle in the force field $\vec{F}(x, y, z) = \langle y + z, x + z, x + y \rangle$ along the line segment from $(1, 0, 0)$ to $(3, 4, 2)$.

   Solution 1: We can find the work done in moving a particle in a force field by evaluating the line integral along the the trajectory followed by the particle.

   To set up the required line integral, we first find a vector function that describes the trajectory of the particle. In our case, the trajectory is a line segment that can be described by $\vec{r}(t) = \langle 1, 0, 0 \rangle + t\langle 3 - 1, 4 - 0, 2 - 0 \rangle = \langle 1 + 2t, 4t, 2t \rangle$, with $0 \leq t \leq 1$.

   From the vector function, we observe that the parametric equations for x, y, and z are $x = 1 + 2t$, $y = 4t$, and $z = 2t$. The forces acting on the particle can be found by evaluating the force field along the particle's trajectory. In our case, $\vec{F}(x(t), y(t), z(t)) = \langle (4t) + (2t), (1 + 2t) + (2t), (1 + 2t) + (4t) \rangle = \langle 6t, 1 + 4t, 1 + 6t \rangle$.

   The next step is to determine $\mathrm{d}\vec{r}$. This can be done by remembering that $\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{r}\,'(t)$. Thus, $\mathrm{d}\vec{r} = \vec{r}\,'(t)\,\mathrm{d}t$. Therefore, we now find $\vec{r}\,'(t) = \langle 2, 4, 2 \rangle$. The line integral then is

   $$W = \int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^1 \langle 6t, 1 + 4t, 1 + 6t \rangle \cdot \langle 2, 4, 2 \rangle\,\mathrm{d}t = \int_0^1 \left[(12t) + (4 + 16t) + (2 + 12t)\right]\,\mathrm{d}t = \int_0^1 (40t + 6)\,\mathrm{d}t$$

   $$W = \left[40\frac{t^2}{2} + 6t\right]_0^1 = (20 + 6) - (0 + 0) = 26.$$

   Note: The solution procedure may be slightly different depending on which form of the line integral you use. However, remember that all those forms are all equivalent.

   Solution 2: This problem can also be solved by the Fundamental Theorem of Calculus for Line Integrals.

First note that the field is defined everywhere in $\mathbb{R}^2$ (This is a necessary but not sufficient condition). Then, if $\vec{F} = \langle P, Q, R\rangle = \nabla f$ for some function f, then $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$, $R = \frac{\partial f}{\partial z}$.

If these assumptions are correct, then $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ should be equal to $\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial P}{\partial z} = \frac{\partial^2 f}{\partial z \partial x}$ should be equal to $\frac{\partial R}{\partial x} = \frac{\partial^2 f}{\partial x \partial z}$, and $\frac{\partial Q}{\partial z} = \frac{\partial^2 f}{\partial z \partial y}$ should be equal to $\frac{\partial R}{\partial y} = \frac{\partial^2 f}{\partial y \partial z}$.

Let us now check whether these conditions are met.
$\frac{\partial P}{\partial y} = 1$, and $\frac{\partial Q}{\partial x} = 1$.
$\frac{\partial P}{\partial z} = 1$, and $\frac{\partial R}{\partial x} = 1$.
$\frac{\partial Q}{\partial z} = 1$, and $\frac{\partial R}{\partial y} = 1$.

Therefore, given that the field is defined everywhere in $\mathbb{R}^2$ and all the mixed-variables second order derivatives are equal, we can be sure that $\vec{F}$ is a gradient field, that is, $\vec{F} = \nabla f$.

Since we now know that $\vec{F} = \nabla f$, then the line integral $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$ can be calculated as $f(P_{final}) - f(P_{initial})$, where $P_{initial} = (1, 0, 0)$ and $P_{final} = (3, 4, 2)$.

We now need to find a function $f(x, y, z)$ such that $\nabla f(x, y, z) = \langle y + z, x + z, x + y\rangle$.

We first integrate $y + z$ with respect to x. This operation results in a candidate $f = xy + xz + C(y, z)$. Now, differentiating with respect to y we can compare with the second term of the vector function. We obtain $x + C_y(y, z)$. By comparison, we conclude that $C_y(y, z) = z$. Therefore $C(y, z) = zy + W(z)$. The new candidate function $f = xy + xz + yz + W(z)$. If we now differentiate this candidate function with respect to z, we obtain $x + y + W_z(z)$. By comparing with the third component of the vector field, we conclude that $W_z(z) = 0$, which means that $W(z) = K$. The final result is then $f(x, y, z) = xy + xz + yz + K$.

We finish by evaluating the function at the end points: $\int_C \vec{F} \cdot \mathrm{d}\vec{r} = f(P_{final}) - f(P_{initial}) = f(3, 4, 2) - f(1, 0, 0) = 3(4) + 3(2) + 4(2) - (0) = 12 + 6 + 8 = 26$, which is the result that we obtained by solving the line integral from first principles.

3. (25 pts) Show that the vector field $\vec{F}(x, y) = y^2 e^{xy}\hat{\imath} + (e^{xy} + xye^{xy})\hat{\jmath}$ is a conservative vector field. Then find a function f such that $\vec{F}(x, y) = \nabla f$.

Solution: First of all, we note that $\vec{F}(x, y)$ is defined everywhere in $\mathbb{R}^2$. This means that if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, then we can say that $\vec{F}(x, y)$ is a conservative field.

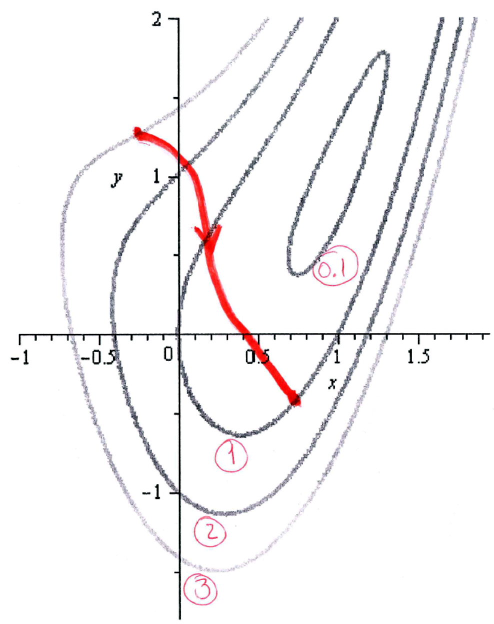
In our case, $P = y^2 e^{xy}$ and $Q = e^{xy} + xye^{xy}$. Then, $\frac{\partial P}{\partial y} = 2ye^{xy} + y^2 e^{xy}(x) = 2ye^{xy} + y^2 xe^{xy}$, and $\frac{\partial Q}{\partial x} = e^{xy}(y) + (ye^{xy} + xye^{xy}(y)) = 2ye^{xy} + y^2 xe^{xy}$. Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, we conclude that $\vec{F}(x, y)$ is a conservative field.

Let us now find a function f such that $\vec{F}(x, y) = \nabla f$. First, we integrate P with respect to x (remember that y is regarded as a constant): $\int P\,\mathrm{d}x = \int y^2 e^{xy}\,\mathrm{d}x$. By using $u = xy$, then $\mathrm{d}u = y\mathrm{d}x$. Therefore, $\int y^2 e^{xy}\,\mathrm{d}x = y \int e^u\,\mathrm{d}u = ye^u + C(y) = ye^{xy} + C(y)$.

We now differentiate $ye^{xy} + C(y)$ with respect to y in order to match terms with the second component of the vector field. If we do that, $\frac{\partial(ye^{xy} + C(y))}{\partial y} = e^{xy} + ye^{xy}(x) + C_y(y) = e^{xy} + xye^{xy} + C_y(y)$. By comparing this result with the second component of the vector field, we find that $C_y(y) = 0$. This means that $C(y) = K$, where K is a constant.

We conclude that $f(x, y) = ye^{xy} + K$.

4. (25 pts) The figure shows a curve C and a contour map of the Rosenbrock function $f(x, y) = (1-x)^2 + (y-x^2)^2$ whose gradient is continuous. The circled numbers are the values of the function along the contour lines. Find $\int_C \nabla f \cdot \mathrm{d}\vec{r}$.



Solution: Since ∇f is a conservative field, we can use the Fundamental Theorem of Calculus for Line Integrals. Therefore, $\int_C \nabla f \cdot \mathrm{d}\vec{r} = f(P_{final}) - f(P_{initial}) = 1 - 3 = -2$

Bonus (10 pts): State Green's Theorem and explain how you could use it to find the area of regions enclosed by simple positively defined closed curves.

Solution: Green's Theorem says that if C is a simple, positively defined closed curve and D is the region bounded by C, then

$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\,\mathrm{d}A.$

One could use Green's Theorem to find the area of regions enclosed by simple positively defined closed curves by finding functions P and Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. For example, $P = -y$ and $Q = 0$, would satisfy the criterion. Then, $\text{Area}(D) = \iint_D \mathrm{d}A = \oint_C P\,\mathrm{d}x = \oint_C -y\,\mathrm{d}x$.